Introduction to Decision Theory

Lecture Notes

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These notes are intended to support an introductory course on decision theory. They contain the mathematical material, including definitions, results, and proofs, but little background, motivation, and discussion. Exercises appear at the end of each chapter. Solutions are available from the author upon request. Covered topics include choice theory, utility representations, expected and subjective expected utility, and ambiguity aversion. In particular, the focus is on decision-making under risk and uncertainty. The approach to most topics is the axiomatic method. Inevitably, many subjects have been left out.

Much of the material in these notes appears in one of the following books. They have been an invaluable source and are warmly recommended as accompanying material for this course and on their own.

- D. M. Kreps. *Notes on the theory of choice.* Underground Classics in Economics. Westview Press, 1988
- I. Gilboa. *Theory of Decision under Uncertainty.* Cambridge University Press, 2009
- K. Binmore. *Rational Decisions.* The Gorman Lectures in Economics. Princeton University Press, 2009
- I. Gilboa. *Rational Choice.* MIT Press, 2010

Future iterations of these notes will benefit from your comments.

1 Relations

This section introduces basic concepts for relations that will be used throughout these notes. For now, X is an arbitrary set.

1.1 Basic Definitions

A binary relation $\succsim$ on X is a set of ordered pairs of elements of X. That is, $\succsim$ is a subset of $X \times X$. For $x, y \in X$, we write $x \succsim y$ if and only if $(x, y) \in \succsim$. Then, $\succsim$ is

(i) reflexive if for all $x \in X$, $x \succsim x$,

(ii) complete if for all $x, y \in X$, either $x \succsim y$ or $y \succsim x$,

(iii) symmetric if for all $x, y \in X$, $x \succsim y$ implies $y \succsim x$,

(iv) asymmetric if for all $x, y \in X$, $x \succsim y$ implies not $y \succsim x$, and

(v) anti-symmetric if for all $x, y \in X$, $x \succsim y$ and $y \succsim x$ implies $x = y$.

The symmetric part of $\succsim$, commonly denoted by $\sim$, is the set of pairs $\{(x, y) \in X \times X : x \succsim y \text{ and } y \succsim x\}$. The asymmetric part of $\succsim$, commonly denoted by $\succ$, is the set of pairs $\{(x, y) \in X \times X : x \succsim y \text{ and not } y \succsim x\}$. One can check from the definitions that $\sim$ is symmetric and $\succ$ is asymmetric.

Example 1.1 (Properties of relations)**.** Let $X = \{a, b, c\}$ and

$$\succsim = \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (b, c)\}.$$

Then, $a \sim a$, $b \sim b$, $c \sim c$, $a \sim b$, $a \succ c$, and $b \succ c$. Hence, $\succsim$ is complete (and thus reflexive) but not symmetric, asymmetric, or anti-symmetric. The symmetric and asymmetric parts are $\sim = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ and $\succ = \{(a, c), (b, c)\}$. In particular, $\sim$ is symmetric, and $\succ$ is asymmetric.

The upper contour set of $x \in X$ is the subset $U_{\succsim}(x)$ of elements of X that are greater than x according to $\succsim$. Similarly, the lower contour set of x is the subset $L_{\succsim}(x)$ of elements of X that are smaller than x.

$$U_{\succsim}(x) = \{y \in X : y \succ x\} \text{ and } L_{\succsim}(x) = \{y \in X : x \succ y\}.$$

If $\succsim$ is clear from the context, we write $U(x)$ and $L(x)$ for short.

For a subset Y of X, we write $\max_{\succsim} Y = \{x \in Y : U(x) \cap Y = \emptyset\}$ and $\min_{\succsim} Y = \{x \in Y : L(x) \cap Y = \emptyset\}$ for the sets of maximal and minimal elements of $\succsim$ in Y, respectively. Note that $\max_{\succsim} Y$ and $\min_{\succsim} Y$ may be empty.

When interpreting relations as an agent's preferences, it is common to use the following terminology.

$$x \succsim y\colon \text{“weak preference for } x \text{ over } y\text{”}$$
$$x \succ y\colon \text{“strict preference for } x \text{ over } y\text{”}$$
$$x \sim y\colon \text{“indifference between } x \text{ and } y\text{”}$$

1.2 Transitivity and Equivalence

The preferences of a rational agent are typically assumed to satisfy some notion of transitivity.

Definition 1.2 (Notions of transitivity)**.** Let $\succsim$ be a relation on X. Then, $\succsim$ is

transitive if for all $x, y, z \in X$, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$, (transitivity)

quasi-transitive if for all $x, y, z \in X$, $x \succ y$ and $y \succ z$ implies $x \succ z$, and (quasi-transitivity)

acyclic if for all $x_1, \dots, x_n \in X$, $x_1 \succ x_2, \dots, x_{n-1} \succ x_n$ implies not $x_n \succ x_1$. (acyclicity)

Quasi-transitivity is transitivity of the asymmetric part of $\succsim$. Acyclicity requires that the asymmetric part of $\succsim$ has no cycles. In fact, the three notions form a hierarchy. Hence, the asymmetric part of a transitive relation is transitive. A short argument shows that the same is true for the symmetric part (Exercise 1.1).

Lemma 1.3. *Transitivity implies quasi-transitivity, and quasi-transitivity implies acyclicity.*

Proof. Exercise. □

To see why preferences that violate, say, acyclicity may be considered irrational, consider the following example. There are three items x, y, z, and an agent who owns one of them, say, x. Suppose further that the agent's preferences are $x \succ y \succ z \succ x$. This agent would prefer to swap x for z, then swap z for y, only to swap y for x, completing the cycle. Thus, the agent would not be able to make a choice for one of the items that she does not wish to revise. The problem becomes more severe if we further assume that the agent prefers making each of these swaps even if she has to pay a tiny amount of money for them. Then, by repeating the above cycle sufficiently often, the agent could be made to pay an arbitrarily large amount of money only to end up with the same item she started with. Similar examples can be constructed for violations of quasi-transitivity and transitivity.

This failure to make a satisfactory choice and the “money pump” argument are often given as the main reasons for insisting on the transitivity of preferences. Section 2 and Section 3 discuss the ramifications of transitivity further. For now, we prove that for acyclic relations, finite sets admit maximal elements, which shows that a satisfactory choice always exists in that case. Note that this does not hold for infinite sets. For example, the natural order on $\mathbb{Z}$ is even transitive but fails to have a maximal element (since there is no largest integer).

Lemma 1.4 (Maximal elements and acyclicity)**.** *Let $\succsim$ be a relation on X. Then, $\succsim$ is acyclic if and only if* $\max_{\succsim} Y \neq \emptyset$ *for every finite subset Y of X.*

Proof. First, assume that $\max_{\succsim} Y \neq \emptyset$ for every finite subset Y of X. We have to prove that $\succsim$ is acyclic. Assume that $x_1 \succ x_2 \succ \ldots \succ x_n$ and let $Y = \{x_1, \ldots, x_n\}$. Since $\max_{\succsim} Y \neq \emptyset$, it follows that $\max_{\succsim} Y = \{x_1\}$. Hence, not $x_n \succ x_1$. This proves that $\succsim$ is acyclic.

Conversely, assume that $\succsim$ is acyclic and let $Y \subset X$ be finite. We must prove that $\max_{\succsim} Y \neq \emptyset$. Let $x_1 \in Y$. If $x_1 \in \max_{\succsim} Y$, we are done. Otherwise, there is $x_2 \in Y$ with $x_2 \succ x_1$. If $x_2 \in \max_{\succsim} Y$, we are done. Otherwise, there is $x_3 \in Y$ with $x_3 \succ x_2$. Iterating this process, we get a sequence $x_1, x_2, x_3, \ldots$ so that $x_{k+1} \succ x_k$ for all k. Since $\succsim$ is acyclic, the x_k are all distinct. So since Y is finite, the sequence terminates with some $x_n \in \max_{\succsim} Y$, and so $\max_{\succsim} Y \neq \emptyset$. $\square$

An important class of relations are equivalence relations, which can be viewed as a generalization of the familiar notion of equality.

Definition 1.5 (Equivalence relations)**.** A relation $\sim$ on X is an equivalence relation if it is reflexive, symmetric, and transitive. The equivalence class of $x \in X$ is $\{y \in X : y \sim x\}$, denoted by $[x]_\sim$. The set of equivalence classes of $\sim$ is denoted by $X/{\sim}$ (speak "X modulo $\sim$").

The following lemma makes precise in which sense equivalence relations generalize equality.

Lemma 1.6 (Quotient maps)**.** *A relation $\sim$ on X is an equivalence relation if and only if there is a set A and a function $f\colon X \to A$ so that for all $x, y \in X$,*

$$x \sim y \textit{ if and only if } f(x) = f(y).$$

Proof. If $\sim$ is an equivalence relation, let $A = X/{\sim}$ be the set of equivalence classes of $\sim$ and let $f\colon X \to A$ be the function mapping $x \in X$ to its equivalence class $[x]_\sim$, so that $f(x) = [x]_\sim$. (This function f is often called the quotient map of $\sim$.) Then,

$$x \sim y \leftrightarrow [x]_\sim = [y]_\sim \leftrightarrow f(x) = f(y).$$

Conversely, suppose there are A and f as in the statement of the lemma. Then, $\sim$ is an equivalence relation since $=$ is. More explicitly, $\sim$ is reflexive since $f(x) = f(x)$ and so $x \sim x$ for all $x \in X$; $\sim$ is symmetric since $f(x) = f(y)$ implies $f(y) = f(x)$ for all $x, y \in X$; $\sim$ is transitive since $f(x) = f(y)$ and $f(y) = f(z)$ implies $f(x) = f(z)$ for all $x, y, z \in X$. $\square$

Example 1.7 (Equivalence relations and quotient maps)**.** Let $X = \{a, b, c\}$ and $\succsim$ be the reflexive relation on X with $a \succ c$, $b \succ c$, and $a \sim b$. Then, the symmetric part $\sim$ of $\succsim$ is an equivalence relation, and $[a]_\sim = [b]_\sim = \{a, b\}$ and $[c]_\sim = \{c\}$. The quotient map $f\colon X \to X/{\sim}$ is given by $f(a) = f(b) = [a]_\sim$ and $f(c) = [c]_\sim$. Note, however, that there are relations whose symmetric part is not an equivalence relation.

Considering equivalence relations can simplify some proofs. For example, if $\succsim$ is a complete and transitive relation, then its symmetric part $\sim$ is an equivalence relation (Exercise 1.4). Define a relation $\hat{\succsim}$ on $X/{\sim}$ by letting $[x]_\sim \mathrel{\hat{\succsim}} [y]_\sim$ if and only if $x \succsim y$. Transitivity of $\succsim$ ensures that $\hat{\succsim}$ is well-defined, that is, does not depend on the choice of representatives x and y from their equivalence classes $[x]_\sim$ and $[y]_\sim$. Then, $\hat{\succsim}$ is a complete, transitive, and *anti-symmetric* relation (Exercise 1.4). The additional anti-symmetry of $\hat{\succsim}$ can make studying its properties more convenient. Statements about $\succsim$ can then be inferred from those about $\hat{\succsim}$.

1.3 Exercises

Exercise 1.1 (Transitivity of the symmetric and asymmetric part). Let $\succsim$ be a relation on X with symmetric part $\sim$ and asymmetric part $\succ$.

(i) Show that transitivity of $\succsim$ implies transitivity of $\sim$ and $\succ$.

(ii) Prove or disprove that if $\succsim$ is complete and $\sim$ and $\succ$ are transitive, then $\succsim$ is transitive.

Exercise 1.2 (Notions of transitivity). Prove that transitivity implies quasi-transitivity and that quasi-transitivity implies acyclicity. Give an example of a relation that is quasi-transitive but not transitive and one that is acyclic but not quasi-transitive.

Exercise 1.3 (Negative transitivity). A relation $\succ$ on X is *negatively transitive* if for all $x, y \in X$,

$$\text{if } x \succ y \text{, then for all } z \in X \text{, either } x \succ z \text{ or } z \succ y \text{ (or both).}$$

Show that if $\succsim$ is a *complete* relation, then

$$\succsim \text{ is transitive if and only if its asymmetric part } \succ \text{ is negatively transitive.}$$

Exercise 1.4 (Quotient relation). Let $\succsim$ be a complete and transitive relation on X and $\sim$ its symmetric part. Prove that $\sim$ is an equivalence relation. Then, define a relation $\hat{\succsim}$ on $X/\!\sim$ by letting

$$[x]_\sim \mathrel{\hat{\succsim}} [y]_\sim \text{ if and only if } x \succsim y.$$

Show that $\hat{\succsim}$ is well-defined, complete, transitive, and anti-symmetric.

2 Choice Theory

Let X be a set, interpreted as a set of possible outcomes. The set of menus $\mathcal{M}(X) = \{Y \subset X : Y \text{ is non-empty and finite}\}$ consists of all non-empty and finite subsets of X. A choice function is a function $C : \mathcal{M}(X) \to \mathcal{M}(X)$ so that $C(Y) \subset Y$ for all $Y \in \mathcal{M}(X)$. Note that a choice function has to select the single outcome in each one-element menu and may choose multiple outcomes from larger menus.

Some choice functions do not comply with our intuitive understanding of rationality. For example, for $X = \{a, b, c\}$, suppose that $C(\{a, b, c\}) = \{a\}$ and $C(\{a, b\}) = \{b\}$. The first choice indicates that a is the most preferred outcome among all three outcomes. But the second choice suggests that b is better than a. Hence, whether or not c is present influences the desirability of a compared to b.

The following two subsections examine the connection between choice functions and preferences and formalize consistent choice behavior that avoids unintuitive choices as the one above.

2.1 Rationalizability

Some choice functions arise from choosing maximal elements according to a relation. In that case, we say that the relation rationalizes the choice function. Since choice functions select a non-empty set from each menu, only relations with a non-empty set of maximal elements in each menu can be rationalizing. By Lemma 1.4, these are precisely the acyclic relations. The stronger notion of transitive rationalizability requires the rationalizing relation to be transitive.

Definition 2.1 ((Transitive) rationalizability)**.** A choice function C is (transitively) rationalizable if there exists a (complete and transitive) relation $\succsim$ on X so that $C(Y) = \max_{\succsim} Y$ for all $Y \in \mathcal{M}(X)$.

Example 2.2. Let $X = \{a, b, c\}$ and consider the following choice function C on X. (We omit the set brackets and one-outcome menus for convenience.)

Y	$C(Y)$
ab	ab
ac	a
bc	bc
abc	ab

Then, C is rationalizable by the relation $\succsim$ with $a \sim b$, $b \sim c$, and $a \succ c$. Note that $\succsim$ is complete but not transitive. On the other hand, C is also rationalizable by $\hat{\succsim}$ with $a \mathrel{\hat{\succ}} c$ and all other alternatives being incomparable. Then, $\hat{\succsim}$ is transitive but not complete. In fact, C is not transitively rationalizable.

For an acyclic relation $\succsim$, define the choice function $C_{\succsim}$ with $C_{\succsim}(Y) = \max_{\succsim} Y$ for all $Y \in \mathcal{M}(X)$. It is clear from the definition that $\succsim$ rationalizes $C_{\succsim}$. As noted above, if $\succsim$ is not acyclic, it cannot rationalize any choice function. We aim for a converse to these assertions.

That is, how can we determine if a choice function is rationalizable, and if so, how can we find a rationalizing relation? A crude way of doing this is to check for each (acyclic) relation whether it rationalizes the choice function. But the number of relations is huge, rendering this approach useless in practice.[1] For a better answer, it is useful to consider the base relation of a choice function.

Definition 2.3 (Base relation)**.** The base relation of a choice function C, denoted by $\succsim_C$, is defined as follows. For all $x, y \in X$,

$$x \succsim_C y \text{ if and only if } x \in C(\{x, y\}).$$

Example 2.4. The base relation $\succsim_C$ of the choice function C in Example 2.2 is given by $a \sim_C b$, $b \sim_C c$, and $a \succ_C c$. Hence, it coincides with the first rationalizing relation given above.

Observe that the base relation is necessarily complete. With the preceding definitions, we have natural mappings from the set of acyclic relations to the set of rationalizable choice functions and vice versa. The first maps an acyclic relation $\succsim$ to its induced choice function $C_{\succsim}$. The one in the converse direction maps a rationalizable choice function C to its base relation $\succsim_C$. The composition $C \mapsto \succsim_C \mapsto C_{\succsim_C}$ is the identity, so one can recover each rationalizable choice function from its base relation. However, as seen in Example 2.2, the mapping $\succsim \mapsto C_{\succsim}$ from left to right is not injective, and so an acyclic relation cannot be recovered from the choice function it induces. But it almost can: the composition $\succsim \mapsto C_{\succsim} \mapsto \succsim_{C_{\succsim}}$ maps each acyclic relation to the relation obtained by declaring indifference between any two incomparable outcomes, and thus produces a complete relation. Hence, $\succsim \mapsto C_{\succsim}$ and $C \mapsto \succsim_C$ are inverse maps between the set of complete, acyclic relations and the set of rationalizable choice functions.

$$\{\text{acyclic relations}\} \underset{\succsim \,\mapsto\, C_{\succsim}}{\overset{C \,\mapsto\, \succsim_C}{\rightleftarrows}} \{\text{rationalizable choice functions}\}$$

The relevance of the base relation is that a choice function is rationalizable if and only if it is rationalized by its base relation.

Lemma 2.5 (Rationalizability and base relation)**.** *Let C be a choice function. Then, C is rationalizable if and only if C is rationalized by its base relation.*

Proof. Let $\succsim$ be a relation rationalizing C. Then, for all $x, y \in X$,

$$x \succ y \leftrightarrow \{x\} = \max_{\succsim}\{x, y\} \leftrightarrow \{x\} = C(\{x, y\}) \leftrightarrow x \succ_C y.$$

Hence, $\succ \,=\, \succ_C$. Thus, since maximal elements of a relation are determined by its asymmetric part, $C(Y) = \max_{\succsim} Y = \max_{\succsim_C} Y$ for all $Y \in \mathcal{M}(X)$, so that $\succsim_C$ rationalizes C. The converse is immediate. $\square$

[1] For each pair of outcomes, there are three possible preferences between them (strict preference for the first, strict preference for the second, and indifference). So, if there are n outcomes, there are $3^{\binom{n}{2}}$ possible preference relations.

2.2 Choice Consistency

The goal of this section is to reduce the (transitive) rationalizability of a choice function (that is, the existence of a (complete and transitive) rationalizing relation) to choice consistency properties. These conditions require the choices from different menus to be consistent with each other and, thus, restrict the choice from a menu based on the choices from other menus. Out of the vast zoo of choice consistency conditions considered in the literature, we study three that are sufficient to characterize rationalizability and transitive rationalizability.

Definition 2.6 (Contraction and (strong) expansion)**.** Let C be a choice function.

For all $Y, Z \in \mathcal{M}(X)$, $Z \subset Y$ implies $C(Y) \cap Z \subset C(Z)$. (contraction or α)

For all $Y, Z \in \mathcal{M}(X)$, $C(Y) \cap C(Z) \subset C(Y \cup Z)$. (expansion or γ)

For all $Y, Z \in \mathcal{M}(X)$, $Z \subset Y$ and $C(Y) \cap Z \neq \emptyset$ implies $C(Z) \subset C(Y)$.
(strong expansion or β^+)

One may describe these conditions verbally as follows. Contraction requires that if x is among the best outcomes in some menu, then x is among the best outcomes in any smaller menu containing it. Expansion states that if x is among the best outcomes in two menus, it should also be among the best outcomes in the union of both menus. Lastly, strong expansion stipulates that if a menu contains one of the best outcomes of a larger menu, then the best outcomes of the smaller menu are also among the best outcomes in the larger menu. As the terminology suggests, strong expansion implies expansion (Exercise 2.3). Otherwise, the three conditions are logically independent.

Example 2.7. Let $X = \{a, b, c\}$ and consider the following choice functions.

Y	$C(Y)$
ab	ab
bc	bc
ac	ac
abc	a

Y	$\hat{C}(Y)$
ab	a
bc	b
ac	c
abc	abc

One checks that C satisfies contraction. However, C violates expansion since for $Y = \{a, b\}$ and $Z = \{b, c\}$, we have $b \in C(Y) \cap C(Z)$ but $b \notin \{a\} = C(Y \cup Z)$. On the other hand, $\hat{C}$ satisfies strong expansion. However, $\hat{C}$ violates contraction since for $Y = \{a, b, c\}$ and $Z = \{a, b\}$, we have $b \in \{a, b\} = \hat{C}(Y) \cap Z$ but $b \notin \{a\} = \hat{C}(Z)$.

The conjunction of contraction and strong expansion is known as the weak axiom of revealed preference (WARP) (Samuelson, 1938).

Definition 2.8 (Weak axiom of revealed preference)**.** A choice function C satisfies the weak axiom of revealed preference if for all $Y, Z \in \mathcal{M}(X)$,

$$Z \subset Y \text{ and } C(Y) \cap Z \neq \emptyset \text{ implies } C(Z) = C(Y) \cap Z. \qquad \text{(WARP)}$$

Note that contraction and strong expansion give the set inclusion from right to left and left to right, respectively, on the right-hand side in the definition of WARP. Exercise 2.5 gives an equivalent formulation of WARP, which is convenient for comparing WARP to the strong axiom of revealed preference (SARP) introduced in Exercise 2.6.

Remark 2.9 (Converse pairings of choice consistency conditions). The syntax suggests that strong expansion is a natural converse to contraction. However, the following equivalent formulation of contraction (Exercise 2.3) and a trivial rewriting of expansion make expansion look like a natural converse to contraction.

$$\text{For all } Y, Z \in \mathcal{M}(X),\ C(Y \cup Z) \cap (Y \cap Z) \subset C(Y) \cap C(Z), \text{ and} \qquad \text{(contraction)}$$
$$\text{For all } Y, Z \in \mathcal{M}(X),\ C(Y) \cap C(Z) \subset C(Y \cup Z) \cap (Y \cap Z). \qquad \text{(expansion)}$$

If a choice function is rationalizable, it satisfies contraction since every maximal element of some menu is also a maximal element of every smaller menu containing it. We now show that if a choice function satisfies contraction, its base relation is acyclic. Combining both statements reveals that only choice functions with an acyclic base relation can be rationalizable.

Lemma 2.10 (Contraction and the base relation)**.** *If a choice function C satisfies contraction, then its base relation $\succsim_C$ is acyclic.*

Proof. Assume that $x_1 \succ_C \dots \succ_C x_n$ and suppose that $x_k \in C(\{x_1, \dots, x_n\})$. If $k = 1$, contraction implies that $x_1 \in C(\{x_1, x_n\})$, and so $x_1 \succsim_C x_n$, which proves the asserted acyclicity of $\succsim_C$. If $k > 1$, contraction implies that $x_k \in C(\{x_{k-1}, x_k\})$. But then $x_k \succsim_C x_{k-1}$ by definition of the base relation. Since this contradicts $x_{k-1} \succ_C x_k$, this case cannot occur. $\square$

While contraction is necessary for rationalizability, it is not sufficient. The choice function C in Example 2.7 satisfies contraction. If C were rationalizable, it would be rationalized by its base relation $\succsim_C$ by Lemma 2.5. But $a \sim_C b \sim_C c \sim_C a$, and so $\max_{\succsim_C}\{a, b, c\} = \{a, b, c\} \neq \{a\} = C(\{a, b, c\})$. Hence, C is not rationalizable. Similarly, strong expansion does not guarantee rationalizability (Exercise 2.2). The following theorem shows that, however, the conjunction of contraction and expansion is equivalent to rationalizability. Later, we will see that the conjunction of contraction and strong expansion is equivalent to transitive rationalizability.

Theorem 2.11 (Rationalizability and choice consistency, Sen, 1971)**.** *A choice function C is rationalizable if and only if it satisfies contraction and expansion.*

Proof. Assume that C is rationalizable and let $\succsim$ be a rationalizing relation. To prove contraction, let $Y, Z \in \mathcal{M}(X)$ with $Z \subset Y$. For all $x \in C(Y) \cap Z$, we have $x \in (\max_{\succsim} Y) \cap Z \subset \max_{\succsim} Z = C(Z)$. Hence, $C(Y) \cap Z \subset C(Z)$. To prove expansion, let $Y, Z \in \mathcal{M}(X)$. For all $x \in C(Y) \cap C(Z)$, we have $x \in \max_{\succsim} Y \cap \max_{\succsim} Z \subset \max_{\succsim}(Y \cup Z) = C(Y \cup Z)$. Hence, $C(Y) \cap C(Z) \subset C(Y \cup Z)$.

Conversely, assume that C satisfies contraction and expansion. We show that $\succsim_C$ rationalizes C. That is, for all $Y \in \mathcal{M}(X)$,

$$C(Y) = \max_{\succsim_C} Y.$$

To this end, let $Y \in \mathcal{M}(X)$. Let $x \in C(Y)$. By contraction, $x \in C(\{x, y\})$ and so $x \succsim_C y$ for all $y \in Y$. Hence, $x \in \max_{\succsim_C} Y$. This shows that $C(Y) \subset \max_{\succsim_C} Y$. Conversely, let $x \in \max_{\succsim_C} Y$. By definition of $\succsim_C$, $x \in C(\{x, y\})$ for all $y \in Y$. Applying expansion $|Y| - 2$ times gives that $x \in C(Y)$. This shows that $\max_{\succsim_C} Y \subset C(Y)$. □

Some choice functions are rationalizable by an acyclic relation but not by a complete and transitive relation. For example, the following choice function is rationalizable (since it satisfies contraction and expansion) but is not transitively rationalizable.

Y	$C(Y)$
ab	a
bc	b
ac	ac
abc	a

Observe that C violates strong expansion since $C(\{a, b, c\}) \cap \{a, c\} = \{a\} \neq \emptyset$ but $C(\{a, c\}) = \{a, c\} \not\subset \{a\} = C(\{a, b, c\})$. The following theorem shows that violations of strong expansion are, in fact, the only additional obstruction to transitive rationalizability.

Theorem 2.12 (Transitive rationalizability and choice consistency, Arrow, 1959)**.** *A choice function C is transitively rationalizable if and only if it satisfies contraction and strong expansion. In that case, the base relation $\succsim_C$ is a complete and transitive rationalizing relation.*

Proof. Assume that C is transitively rationalizable and let $\succsim$ be a complete and transitive rationalizing relation. It follows from Theorem 2.11 that C satisfies contraction. To prove strong expansion, let $Y, Z \in \mathcal{M}(X)$ with $Z \subset Y$ and $C(Y) \cap Z \neq \emptyset$. Let $x \in C(Z)$. Since $\succsim$ rationalizes C, we have $x \in \max_{\succsim} Z$. Thus, since $\succsim$ is complete, $x \succsim z$ for all $z \in Z$. In particular, letting $y \in C(Y) \cap Z$, we have $x \succsim y$. Moreover, $y \in \max_{\succsim} Y$, and so $y \succsim z$ for all $z \in Y$ again since $\succsim$ is complete and rationalizes C. Hence, since $\succsim$ is transitive, for all $z \in Y$, $x \succsim z$. Thus, $x \in \max_{\succsim} Y = C(Y)$, which shows that $C(Z) \subset C(Y)$.

Conversely, assume that C satisfies contraction and strong expansion. By Theorem 2.11 and Lemma 2.5, C is rationalized by it base relation $\succsim_C$. The base relation is always complete. To prove that $\succsim_C$ is transitive, let $x, y, z \in X$ with $x \succsim_C y \succsim_C z$. We claim that $x \in C(\{x, y, z\})$. Since $C(\{x, y, z\}) \neq \emptyset$, this claim follows if we can show that $z \in C(\{x, y, z\})$ implies $y \in C(\{x, y, z\})$ and $y \in C(\{x, y, z\})$ implies $x \in C(\{x, y, z\})$. First, if $z \in C(\{x, y, z\})$, we use strong expansion with $Y = \{x, y, z\}$ and $Z = \{y, z\}$. It follows that $y \in C(\{y, z\}) \subset C(\{x, y, z\})$. Second, if $y \in C(\{x, y, z\})$, we use strong expansion with $Y = \{x, y, z\}$ and $Z = \{x, y\}$. It follows that $x \in C(\{x, y\}) \subset C(\{x, y, z\})$. Now, from $x \in C(\{x, y, z\})$, it follows by contraction that $x \in C(\{x, z\})$, and so $x \succsim_C z$. Hence, $\succsim_C$ is transitive. □

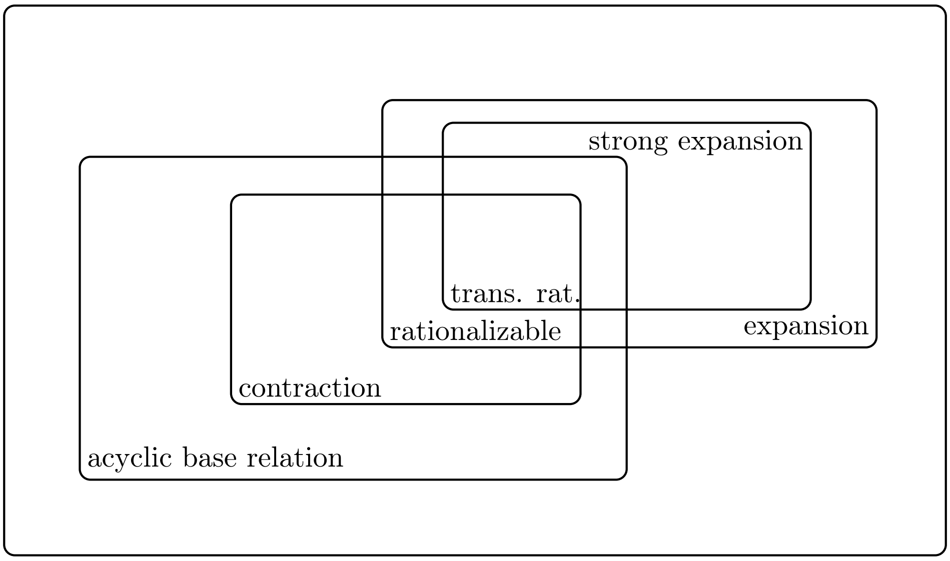


Figure 2.1: Overview of choice consistency and rationalizability conditions. Each rectangle represents the set of choice functions satisfying the corresponding property. The shaded areas denote the set of rationalizable and transitively rationalizable choice functions.

The results of this section are summarized in Figure 2.1.

Remark 2.13 (Quasi-transitive rationalizability). It is natural to ask if rationalizability by a quasi-transitive relation is equivalent to the conjunction of contraction and some notion of expansion consistency. Schwartz (1976) showed that the answer is affirmative but less clean than for rationalizability and transitive rationalizability: a choice function is rationalizable by a quasi-transitive relation if and only if it satisfies contraction (α), expansion (γ), and ε^+.

$$\text{For all } Y, Z \in \mathcal{M}(X),\ C(Y) \subset Z \subset Y \text{ implies } C(Z) \subset C(Y). \qquad (\varepsilon^+)$$

In words, ϵ^+ requires that if a menu contains all of the best outcomes of a larger menu, then the best outcomes of the smaller menu are also among the best outcomes of the larger menu. Observe that ϵ^+ is weaker than strong expansion (β^+) since the latter only requires that $C(Y) \cap Z \neq \emptyset$ in the antecedent (which corresponds to replacing “all” by “some” in the preceding sentence).

In the following, a *preference relation* is always a complete and transitive relation. Theorem 2.12 shows that an agent’s choices satisfy WARP if and only if they are as if the agent was choosing maximal elements according to a preference relation. Transitivity and completeness are often viewed as the cornerstones of rational decision-making. While transitivity is normatively appealing, it is not perfectly descriptive since decision-makers in experiments sometimes violate transitivity. However, this may be due to cognitive limitations. If an intransitivity was pointed out to them, they may wish to revise their choices to eliminate the intransitivity. The following quote of Savage (1954, p. 21) speaks to that.

> “[...] when it is explicitly brought to my attention that I have shown a preference for f as compared with g, for g as compared with h, and for h as compared with

f, I feel uncomfortable in much the same way that I do when it is brought to my attention that some of my beliefs are logically contradictory."

2.3 Exercises

Exercise 2.1 (Trivial cases of choice consistency conditions). Show that the implications in the definitions of α and β^+ hold for any choice function if the smaller of the two menus contains only one outcome. That is, show that for any choice function C, the following holds: for all $Y, Z \in \mathcal{M}(X)$ with $|Z| = 1$, $Z \subset Y$ implies $C(Y) \cap Z \subset C(Z)$. Prove the analogous statement for β^+. Also, show that the implication in the definition of γ holds for any choice function if one of the menus is contained in the other.

Exercise 2.2 (Properties of choice functions). Consider the following choice functions C and $\hat{C}$.

Y	$C(Y)$
$\{a, b\}$	$\{b\}$
$\{a, c\}$	$\{a\}$
$\{b, c\}$	$\{b, c\}$
$\{a, b, c\}$	$\{b\}$

Y	$\hat{C}(Y)$
$\{a, b\}$	$\{a, b\}$
$\{a, c\}$	$\{a\}$
$\{a, d\}$	$\{a\}$
$\{b, c\}$	$\{b, c\}$
$\{b, d\}$	$\{b\}$
$\{c, d\}$	$\{c\}$
$\{a, b, c\}$	$\{a, b\}$
$\{a, b, d\}$	$\{a, b\}$
$\{a, c, d\}$	$\{a\}$
$\{b, c, d\}$	$\{b\}$
$\{a, b, c, d\}$	$\{a, b\}$

(i) Draw the asymmetric parts $\succ_C$ and $\succ_{\hat{C}}$ of the base relations $\succsim_C$ and $\succsim_{\hat{C}}$.

(ii) Check whether C and $\hat{C}$ satisfy α and γ.

(iii) Are C and $\hat{C}$ rationalizable? Why?

(iv) Give an example of a choice function that satisfies β^+ but is not rationalizable.

Exercise 2.3 (Implications between choice consistency conditions). Prove the following.

(i) α is equivalent to the following condition (cf. Remark 2.9).

$$\text{For all } Y, Z \in \mathcal{M}(X),\ C(Y \cup Z) \cap (Y \cap Z) \subset C(Y) \cap C(Z).$$

(ii) β^+ implies γ.

Exercise 2.4 (Quasi-transitive rationalizability). Give an example of a choice function that is rationalizable but not rationalizable by a quasi-transitive relation.

Exercise 2.5 (Equivalent formulation of WARP). Prove that a choice function C satisfies WARP if and only if for all $Y, Z \in \mathcal{M}(X)$ and $x, y \in Y \cap Z$,

$$x \in C(Y) \text{ and } y \in C(Z) \text{ implies } x \in C(Z).$$

Exercise 2.6 (SARP). A choice function C satisfies the strong axiom of revealed preference (SARP) if for all $x_1, \dots, x_n \in X$ and $X_1, \dots, X_n \in \mathcal{M}(X)$,

$$\begin{aligned} &x_i, \dots, x_n \in X_i,\ x_i \in C(X_i) \text{ for all } i = 1, \dots, n \text{ and } x_1, \dots, x_n \in X_n \\ &\text{implies } x_1, \dots, x_{n-1} \in C(X_n). \end{aligned} \tag{SARP}$$

Show that WARP is equivalent to SARP. (Note: Sen (1971) has shown that WARP and SARP are equivalent for all choice functions that are defined on all menus (with at least three outcomes) since then WARP implies transitivity of the rationalizing relation. Social scientists do, however, typically not have the resources to observe agents' choices from all menus. To infer transitive preferences from choices from pairs of outcomes, they restrict themselves to choice function satisfying SARP. For choice functions that are defined only for pairs of outcomes (and are thus not choice functions under our definition), SARP is stronger than WARP. For instance, in that case, SARP implies that if x is chosen when y is available and y is chosen when z is available, then whenever z is chosen and x is available, x is also chosen, while WARP does not.)

Exercise 2.7 (Resolute choice functions). A choice function C is *resolute* if $|C(Y)| = 1$ for all $Y \in \mathcal{M}(X)$. Show that a resolute choice function C is rationalizable if and only if it satisfies α. In that case, C is rationalizable by a complete, transitive, and anti-symmetric relation.

Exercise 2.8 (Revealed preference relation). Let C be a choice function. The *revealed preference relation* $\tilde{\succsim}_C$ of C on X is defined as follows. For all $x, y \in X$,

$$x \mathrel{\tilde{\succsim}_C} y \text{ if and only if there is some } Y \in \mathcal{M}(X) \text{ such that } x \in C(Y) \text{ and } y \in Y.$$

(i) Let C be a choice function that satisfies α. Show that the revealed preference relation $\tilde{\succsim}_C$ and the base relation $\succsim_C$ coincide.

(ii) Give an example of a choice function C for which $\tilde{\succsim}_C$ and $\succsim_C$ do not coincide.

(iii) Show that if a choice function C satisfies β^+, then $\tilde{\succsim}_C$ is transitive.

(iv) Show that the implication in (iii) does not hold in the opposite direction.

3 Utility Representation

Section 2 studied conditions under which a choice function can be expressed as maximization of a preference relation. Now we consider the problem of representing a preference relation by a utility function, that is, a function $u\colon X \to \mathbb{R}$. Taking both steps together allows one to summarize a choice function by a utility function, which is frequently more amenable to economic reasoning.

Definition 3.1 (Utility representation)**.** A utility function $u\colon X \to \mathbb{R}$ represents a relation $\succsim$ if it is order-preserving. That is, u represents $\succsim$ if for all $x, y \in X$,

$$x \succsim y \text{ if and only if } u(x) \geq u(y).$$

Note that $\succsim$ admits a utility representation if and only if the relation $\hat{\succsim}$ over the equivalence classes $X/{\sim}$ does (see Section 1.2). For suppose $\hat{u}$ represents $\hat{\succsim}$. Then defining u by letting $u(x) = \hat{u}([x]_\sim)$ for all $x \in X$ gives a representation of $\succsim$. Since $\hat{\succsim}$ is anti-symmetric, there is no loss of generality in assuming that $\succsim$ is anti-symmetric, and we shall do so in this section whenever convenient.

It is important to point out that the fact that a relation can be represented by a utility function does not imply that the preferences are derived from some measurable quantity (utility) obtained from the outcomes. Absent further justification, u is best viewed as a mathematical object that is convenient for working with preference relations. The following statement clarifies that utility representations are far from unique and no canonical utility representation exists.

Lemma 3.2 (Increasing transformations)**.** *If a utility function u represents a relation $\succsim$ and $\phi\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $\phi \circ u$ represents $\succsim$.*

Proof. Let $x, y \in X$. Since u represents $\succsim$ and ϕ is strictly increasing, we have

$$x \succsim y \leftrightarrow u(x) \geq u(y) \leftrightarrow (\phi \circ u)(x) \geq (\phi \circ u)(y),$$

and so $\phi \circ u$ represents $\succsim$. □

3.1 Representation of Preference Relations

We now characterize which relations admit a utility representation. Clearly, any such relation must be a preference relation, that is, complete and transitive. Conversely, every preference relation over finitely many or countably infinitely many outcomes can be represented by a utility function. This can be proven as follows. Enumerate all outcomes in some way, say, $X = \{x_1, x_2, x_3, \ldots\}$, and construct a representing utility function u inductively. Let $u(x_1) = 0$. If x_2 is indifferent to x_1, let $u(x_2) = 0$; if x_2 is strictly preferred to x_1, let $u(x_2)$ be larger than 0, say, $u(x_2) = 1$; if x_2 is strictly less preferred than x_1, let $u(x_2)$ be smaller than 0, say, $u(x_2) = -1$. For x_3 a new case arises: if x_3 ranks in between x_1 and x_2, let $u(x_3)$ be some value in between $u(x_1)$ and $u(x_2)$, say, $u(x_3) = \frac{1}{2}u(x_1) + \frac{1}{2}u(x_2)$. If x_3 is indifferent to x_1 or x_2, strictly preferred to both x_1 and x_2, or strictly less preferred than both x_1 and x_2, proceed as in the case of x_2. In general, having defined $u(x_k)$ for all $k \in [n]$,[2] let

$$u(x_{n+1}) = \begin{cases} u(x_k) & \text{if } x_{n+1} \sim x_k \text{ for some } k \in [n], \\ \max_{k\in[n]} u(x_k) + 1 & \text{if } x_{n+1} \succ x_k \text{ for all } k \in [n], \\ \min_{k\in[n]} u(x_k) - 1 & \text{if } x_k \succ x_{n+1} \text{ for all } k \in [n], \text{ and} \\ \frac{1}{2}u(x_k) + \frac{1}{2}u(x_l) & \text{if } x_k \succ x_{n+1} \succ x_l \text{ and } x_k \succ x_m \succ x_l \text{ for no } m \in [n]. \end{cases}$$

For each n, u represents $\succsim$ on the set $\{x_1, \dots, x_n\}$. Hence, continuing this construction gives a utility function representing $\succsim$ on the set of all outcomes X. One can also give an explicit (that is, non-recursive) definition of u in terms of $\succsim$ (cf. Exercise 3.1).

If there are uncountably many outcomes, not every preference relation has a utility representation (Exercise 3.3). As we will see, failures of the separability condition below are the only additional obstruction to admitting a utility representation. Intuitively, it states that countably many outcomes suffice to pin down the preferences. More precisely, there exists a countable set of outcomes so that if one knows for every outcome how it relates to those countably many, one also knows how it relates to all others.

Definition 3.3 (Separability)**.** Let $\succsim$ be a relation on a set X. Then, $\succsim$ is separable if there is a countable subset Z of X so that for all $x, y \in X \setminus Z$ with $x \succ y$, there is $z \in Z$ with $x \succ z \succ y$.

Note that for countable sets of outcomes, *every* relation is separable since one can take $Z = X$. Separability is often called a technical axiom. It is required for the mathematics to work out, but lacks a normative justification or economic interpretation. Moreover, it cannot be empirically refuted since this would require an infinite, even uncountable number of observations. This also means that it always holds empirically.

The following theorem shows that an (anti-symmetric) preference relation admits a utility representation if and only if it is separable. It was originally due to the mathematician Georg Cantor, who was not concerned with utility functions but studied many facets of the real line. He asked when a set with an order on it can be embedded into the reals so that the order is preserved. This is the mathematical formulation of the existence of a utility representation. Since preferences over countably many outcomes are always separable, the theorem implies our earlier observation about that case.

Theorem 3.4 (Utility representation, Cantor, 1915; Kreps, 1988)**.** *Let $\succsim$ be an anti-symmetric relation on a set X. Then, $\succsim$ is complete, transitive, and separable if and only if it admits a utility representation. In that case, u is unique up to strictly increasing transformations.*

Proof. Assume that $\succsim$ is a complete, transitive, and separable relation with separating set $Z \subset X$. Since Z is countable, we may write $Z = \{z_1, z_2, \dots\}$. Define $u\colon X \to \mathbb{R}$ by letting, for $x \in X$,[3]

$$u(x) = \sum_{i\in\mathbb{N}\colon z_i \in L(x)} \frac{1}{2^i} - \sum_{i\in\mathbb{N}\colon z_i\in U(x)} \frac{1}{2^i}.$$

[2]Here and later, we use the shorthand $[n] = \{1, \dots, n\}$.

[3]The numbers 2^{-i} are one of many possible choices. Every summable sequence of positive numbers would do.

To see that u represents $\succsim$, let $x, y \in X$. If $x \sim y$, it follows from transitivity and completeness of $\succsim$ that $U(x) \cap Z = U(y) \cap Z$ and $L(x) \cap Z = L(y) \cap Z$. Hence, $u(x) = u(y)$. If $x \succ y$, then $U(x) \cap Z \subset U(y) \cap Z$ and $L(y) \cap Z \subset L(x) \cap Z$ again since $\succsim$ is complete and transitive. We want to show that at least one of these inclusions is strict. To this end, we distinguish three cases.

(i) If $x, y \in X \setminus Z$, there is $z_i \in Z$ with $x \succ z_i \succ y$ since $\succsim$ is separable and Z is a separating set. Hence, $z_i \in L(x) \cap U(y) \cap Z$, so that both inclusions are strict.

(ii) If $x \in Z$, say, $x = z_i$, then $z_i \in U(y) \setminus U(x)$, so that the first inclusion is strict.

(iii) If $y \in Z$, say, $y = z_j$, then $z_j \in L(x) \setminus L(y)$, so that the second inclusion is strict.

Hence, $u(x) > u(y)$. The case $y \succ x$ is analogous.

Conversely, assume $\succsim$ has a utility representation u. Clearly, $\succsim$ is complete and transitive and hence a preference relation. To show that $\succsim$ is separable, let $I = \{u(x)\colon x \in X\} \subset \mathbb{R}$ be the image of u. We construct a countable subset J of I that separates any two elements of $I \setminus J$. Let $J_1 \subset I$ be countable and dense in I and

$$J_2 = \{s, t \in I\colon s > t \text{ and there is no } r \in I \text{ with } s > r > t\}.$$

By a standard argument, one shows that J_2 is countable. Then, $J = J_1 \cup J_2$ has the desired property. For if $s, t \in I \setminus J$ with $s > t$, then $(t, s) \cap I$ is non-empty and open in I and thus contains some element of J_1.

Now let $Z = u^{-1}(J)$. Since $\succsim$ is anti-symmetric, u is bijective, so Z is countable. Let $x, y \in X \setminus Z$ with $x \succ y$. Then there is $r \in J$ with $u(x) > r > u(y)$. Letting $z \in Z$ with $u(z) = r$ gives $u(x) > u(z) > u(y)$. Since u represents $\succsim$, it follows that $x \succ z \succ y$, which finishes the proof.

For uniqueness, assume that u and v represent $\succsim$. We need to find a strictly increasing function $\phi\colon \mathbb{R} \to \mathbb{R}$ so that $v = \phi \circ u$. It is clear that if $s = u(x)$ and $t = v(x)$ for some $x \in X$, then $\phi(s) = t$ needs to hold. But this defines ϕ only on the image of u, which may not be all of $\mathbb{R}$. The following construction fills in the missing values so that ϕ is strictly increasing.

For $s \in \mathbb{R}$, let

$$\overline{s} = \inf\{u(x)\colon u(x) \geq s\} \text{ and } \underline{s} = \sup\{u(x)\colon u(x) \leq s\}, \text{ and}$$
$$\overline{t} = \inf\{v(x)\colon u(x) \geq s\} \text{ and } \underline{t} = \sup\{v(x)\colon u(x) \leq s\}.$$

Then, for $\lambda \in [0, 1]$ with $s = \lambda\overline{s} + (1 - \lambda)\underline{s}$, let

$$\phi(s) = \lambda\overline{t} + (1 - \lambda)\underline{t}.$$

Note that if $s = u(x)$ and $t = v(x)$ for some $x \in X$, then $s = \overline{s} = \underline{s}$ and $t = \overline{t} = \underline{t}$. Hence, $v(x) = \phi(u(x))$, so that $v = \phi \circ u$. It is, moreover, straightforward to check that ϕ is strictly increasing. $\square$

To motivate the following question, assume there are $d \in \mathbb{N}$ different and arbitrarily divisible commodities, and an outcome is a bundle with some non-negative amount of each commodity. That is, $X = \mathbb{R}^d_+$. Then, it is sometimes desirable to find not only a utility representation of a preference relation on X but one that is *continuous* to ensure that small changes to a bundle only lead to small changes in utility.

More generally, if X is any subset of $\mathbb{R}^d$, we can ask when a relation $\succsim$ on X can be represented by a continuous utility function $u\colon X \to \mathbb{R}$. We call such a u a continuous utility representation. Even if $\succsim$ admits a utility representation, it may not admit a continuous one. To see what can go wrong, let $x \in X$ and observe that if u is continuous,

$$u^{-1}((u(x), \infty)) = \{y \in X \colon u(y) > u(x)\} \text{ and } u^{-1}((-\infty, u(x))) = \{y \in X \colon u(y) < u(x)\}$$

need to be open subsets of X. But if u represents $\succsim$, then $\{y \in X \colon u(y) > u(x)\} = U(x)$ and $\{y \in X \colon u(y) < u(x)\} = L(x)$. Hence, a necessary condition for a relation to admit a continuous utility representation is that all upper and lower contour sets are open. In other words, for any $x, y \in X$ with x being strictly preferred to y, any outcome sufficiently close to x is also preferred to y, and any outcome sufficiently close to y is less preferred than x. The following theorem shows that this is the only additional requirement.

Theorem 3.5 (Continuous utility representation)**.** *Let $X \subset \mathbb{R}^d$ and $\succsim$ be a complete, transitive, and separable relation on X so that $U(x)$ and $L(x)$ are open in X for all $x \in X$. Then, $\succsim$ admits a continuous utility representation u.*

Proof. Throughout the proof, we say that a set is open if it is open in X.[4] By Theorem 3.4 and the remarks after Example 1.7, $\succsim$ admits a utility representation, say, u_0. However, u_0 might have discontinuities. We construct a continuous utility representation of $\succsim$ by removing the discontinuities from u_0.

For $s, t \in \mathbb{R}$ with $s < t$, we say that $\{s, t\}$ is a gap of u_0 if

(i) there is no $x \in X$ with $u_0(x) \in (s, t)$,

(ii) $\sup\{u_0(x) \leq s \colon x \in X\} = s$ and $\inf\{u_0(x) \geq t \colon x \in X\} = t$, and

(iii) either there is $x \in X$ with $u_0(x) = s$ or there is $x \in X$ with $u_0(x) = t$ but not both.

A short argument shows that the number of gaps is countable.[5] So let $\{s_1, t_1\}, \{s_2, t_2\}, \ldots$ with $s_k < t_k$ for all k be an enumeration of all gaps.

Without loss of generality, we may assume that 0 is in the range of u_0. We define $u\colon X \to \mathbb{R}$

[4] $Y \subset X$ is open in X if there is a set Y' that is open in $\mathbb{R}^d$ and $Y = Y' \cap X$.

[5] The intervals given by any two gaps are disjoint except for possibly one endpoint each. That is, if $\{s, t\}$ and $\{s', t'\}$ are gaps, then either $t \leq s'$ or $t' \leq s$. For each $n, N \in \mathbb{N}$, let $G_{n,N} = \{\{s, t\} \subset [-N, N] \colon \{s, t\} \text{ is a gap with } t - s \geq \frac{1}{n}\}$. Each $G_{n,N}$ is finite, and the set of all gaps is $\bigcup_{n,N} G_{n,N}$. Hence, the number of gaps is countable.

as follows. For $x \in X$, let

$$u(x) = \begin{cases} u_0(x) - \sum_{k \geq 1\colon 0 \leq s_k < t_k \leq u_0(x)} t_k - s_k & \text{if } u_0(x) \geq 0, \text{ and} \\ u_0(x) + \sum_{k \geq 1\colon u_0(x) \leq s_k < t_k \leq 0} t_k - s_k & \text{if } u_0(x) < 0. \end{cases}$$

One can verify that $u(x) \geq u(y)$ if and only if $u_0(x) \geq u_0(y)$. Hence, u represents $\succsim$.

We show that u is continuous. That is, for all $x \in X$ and $\epsilon > 0$, there is an open neighborhood W of x so that $u(y) \in (u(x) - \epsilon, u(x) + \epsilon)$ for all $y \in W$. It suffices to find open neighborhoods $\overline{W}$ and $\underline{W}$ of x so that $u(y) < u(x) + \epsilon$ for all $y \in \overline{W}$ and $u(y) > u(x) - \epsilon$ for all $y \in \underline{W}$, since then we can take $W = \overline{W} \cap \underline{W}$. Assume that $u_0(x) \geq 0$ and let $\overline{u} = \inf\{u_0(y) \colon y \succ x\}$. We distinguish two cases.

Case 1. Suppose there is $y \in X$ with $y \succ x$ and $u_0(y) = \overline{u}$. Then, for all $z \in X$ with $y \succ z$, $u_0(z) \leq u_0(x)$, which is equivalent to $u(z) \leq u(x)$. We may thus take $\overline{W} = L(y)$, which is open by assumption.

Case 2. Otherwise, let $y \in X$ with $y \succ x$ and $u_0(y) < \overline{u} + \epsilon$. If $\overline{u} = u_0(x)$, then $u_0(y) < u_0(x) + \epsilon$, so that $u(y) < u(x) + \epsilon$. If $\overline{u} > u_0(x)$, then $\{u_0(x), \overline{u}\}$ is a gap. By construction of u,

$$u(y) - u(x) \leq u_0(y) - u_0(x) - (\overline{u} - u_0(x)) = u_0(y) - \overline{u} < \epsilon.$$

In either case, we get $u(z) < u(x) + \epsilon$ for all $z \in X$ with $y \succ z$, and so we may take $\overline{W} = L(y)$.

The construction of $\underline{W}$ and the case $u_0(x) < 0$ are similar. $\square$

Observe that the proof of Theorem 3.5 at no point uses that X is a subset of $\mathbb{R}^d$. It holds just as well for an arbitrary topological space X. This level of generality is, however, seldom needed for applications. From Theorem 3.5, one can obtain without much work the following result by Debreu (1959) (cf. Exercise 3.5).

Corollary 3.6 (Continuous utility representation for connected sets, Debreu, 1959)**.** *Let $X \subset \mathbb{R}^d$ be connected and $\succsim$ a complete and transitive relation on X so that $U(x)$ and $L(x)$ are open in X for all $x \in X$. Then, $\succsim$ admits a continuous utility representation u.*

3.2 Representation of Semi-Orders: Just Noticeable Differences

Recall that we require preference relations to be complete and transitive. Transitivity of a complete relation is equivalent to transitivity of its symmetric and asymmetric parts (cf. Exercise 1.1) If subjects exhibit preference cycles in experiments, they are sometimes attributed to cognitive limitations. More specifically, intransitivities of the indifference relation can be due to measurement limitations. Luce (1956) gives the following example. A decision-maker has preferences over the amount of sugar in a cup of coffee. However, we cannot expect the decision-maker to be able to discern between, say, n and $n + 1$ milligrams of sugar and thus express indifference between both options. Transitivity of the indifference relation would imply that the decision-maker is indifferent between any amount of sugar and no sugar at all, which will not generally be the case.

This example is an instance of Weber's law in psychophysiology, which states that a person's ability to discern differences in the intensity of stimuli is limited. This gives rise to the concept of a "just noticeable difference", the minimal difference in intensity that allows noticing a difference. Luce (1956, p. 122) expressed this as follows.

> "The nontransitiveness of indifference must be recognized and explained on [...] any theory of choice, and the only explanation that seems to work is based on the imperfect powers of discrimination of the human mind whereby inequalities become recognizable only when of sufficient magnitude."

This section seeks to explain this effect in terms of just noticeable differences in utility. More precisely, we consider relations that possess all properties of preference relations except transitivity of the symmetric part and derive a utility representation for those in terms of just noticeable differences.

Definition 3.7 (Semi-order)**.** A relation $\succsim$ on X is a semi-order if it is complete and for all $x, y, z, w \in X$,

$$x \succ y \succ z \sim w \text{ implies } x \succ w, \text{ and} \tag{SO1}$$

$$x \succ y \sim z \succ w \text{ implies } x \succ w. \tag{SO2}$$

Every preference relation is a semi-order. The converse is not true. For example, if $X = \{x, y, z\}$, then $\succsim$ with $x \sim y \sim z$ and $x \succ z$ is a semi-order but not a preference relation since $\sim$ is not transitive. On the other hand, any semi-order $\succsim$ is quasi-transitive, that is, $\succ$ is transitive. For if $x \succ y \succ z$, then since $z \sim z$, SO1 implies $x \succ z$. On the other hand, not every quasi-transitive relation is a semi-order. For example, if $X = \{x, y, z, w\}$, then $\succsim$ with $x \succ y$, $z \succ w$, and indifference otherwise is quasi-transitive but violates SO2. To sum up, every transitive relation is a semi-order, and every semi-order is quasi-transitive. Both converses fail.

The definition of semi-orders is asymmetric in the sense that it does not postulate that for all $x, y, z, w \in X$,

$$x \sim y \succ z \succ w \text{ implies } x \succ w. \tag{SO3}$$

This is because SO3 follows from SO1. In fact, the following holds (cf. Exercise 3.6).

Lemma 3.8 (Equivalent definitions of semi-orders)**.** *For a complete relation $\succsim$ on X,*

(i) SO1 is equivalent to SO3, and

(ii) SO1 and SO2 are logically independent.

A representation in terms of just noticeable differences consists of a utility function and a threshold value. One outcome is preferred to another if their difference in utility is above the threshold; otherwise, indifference holds. The threshold is the minimal noticeable difference in utility.

Definition 3.9 (Utility semi-representation)**.** Let $\succsim$ be a complete relation on X, $u\colon X \to \mathbb{R}$, and $\delta > 0$. Then, (u, δ) is a utility semi-representation of $\succsim$ if for all $x, y \in X$,

$$x \succ y \text{ if and only if } u(x) - u(y) > \delta.$$

We say that $\succsim$ admits a utility semi-representation if there is a pair (u, δ) that is a utility semi-representation of $\succsim$.

Note that a utility semi-representation with $\delta = 0$ is a utility representation in the sense of Definition 3.1. It is straightforward to check that any relation $\succsim$ that admits a utility semi-representation is a semi-order. To see this, suppose (u, δ) is a utility semi-representation of a complete relation $\succsim$. Let us verify that $\succsim$ satisfies (SO1). If $x \succ y \succ z \sim w$, we have that

$$u(x) - u(y) > \delta \qquad u(y) - u(z) > \delta \qquad \delta \geq u(z) - u(w) \geq -\delta.$$

Summing these three inequalities, we get

$$u(x) - u(w) = (u(x) - u(y)) + (u(y) - u(z)) + (u(z) - u(w)) > \delta + \delta - \delta = \delta,$$

which implies $x \succ w$. (SO2) can be verified similarly. The main result of this section is that any semi-order on a finite set of outcomes admits a utility semi-representation.

Theorem 3.10 (Utility semi-representation of semi-orders, Luce, 1956)**.** *Let X be finite. Then, a relation $\succsim$ on X admits a utility semi-representation if and only if it is a semi-order.*

Remark 3.11 (Luce's theorem fails for infinitely many outcomes). Theorem 3.10 fails if the set of outcomes X is infinite. Let $X = [0, 1] \cap \mathbb{Q}$ and let $\epsilon \in (0, \frac{1}{2}) \cap \mathbb{Q}$. Consider the relation $\succsim$ on X defined as follows: for all $x, y \in X$,

$$x \succsim y \text{ if and only if } x \geq (1 - \epsilon)y.$$

It can be verified that $\succsim$ is a semi-order, but does not admit a utility semi-representation (see Exercise 3.7). I am grateful to Cedric Neumann, a former student of this course, for suggesting this example. To the best of my knowledge, it was previously unknown whether Theorem 3.10 holds if X is infinite.

It is possible that $\succsim$ cannot directly discern between two outcomes x and y, but comparing both to a third outcome, we can infer an implicit preference for x over y. More precisely, suppose $x \sim y$ and there is an outcome z so that either $x \sim z \succ y$ or $x \succ z \sim y$. This is evidence that the utility of x is higher than the utility of y, but the difference is too small to be noticeable. The induced inferred preference relation $\hat{\succsim}$ is transitive and the primary tool for constructing a utility semi-representation of $\succsim$.

Lemma 3.12. *Let $\succsim$ be a semi-order. Define a relation $\hat{\succsim}$ on X by letting for all $x, y \in X$,*

$$x \mathrel{\hat{\succsim}} y \text{ if and only if } U_{\succsim}(x) \subset U_{\succsim}(y) \text{ and } L_{\succsim}(y) \subset L_{\succsim}(x).$$

Then, $\hat{\succsim}$ is complete and transitive. Moreover, $x \succ y$ implies $x \mathrel{\hat{\succ}} y$.

Proof. Transitivity of $\hat{\succsim}$ follows from the transitivity of set inclusion. We prove that $\hat{\succsim}$ is complete. Let $x, y \in X$. If $U(x) = U(y)$ and $L(x) = L(y)$, then $x \hat{\sim} y$. If $U(x) \neq U(y)$, assume without loss of generality $U(y) \not\subset U(x)$ and let $z \in U(y) \setminus U(x)$. First, for $w \in L(y)$, we have $x \succsim z \succ y \succ w$, and so $x \succ w$ by Lemma 3.8 and SO3. We conclude that $L(y) \subset L(x)$. Second, for $w \in U(x)$, we have $w \succ x \succsim z \succ y$, and so $w \succ y$ by SO2. We conclude that $U(x) \subset U(y)$. Hence, $x \hat{\succsim} y$. The case $L(x) \neq L(y)$ is similar.

If $x \succ y$, then $y \in L(x) \setminus L(y)$ and completeness of $\hat{\succsim}$ implies that $x \hat{\succ} y$. □

Proof of Theorem 3.10. Let $X = \{x_1, \dots, x_n\}$ with $x_1 \hat{\succsim} \dots \hat{\succsim} x_n$. We construct (u, δ) inductively. Let $u(x_1) = 0$ and $\delta_0 \geq 0$ be arbitrary. Trivially, (u, δ_0) is a utility semi-representation of $\succsim$ restricted to $\{x_1\}$ (viewing u as a function defined only on $\{x_1\}$). Now let $k > 1$ and assume we have defined u on $\{x_1, \dots, x_{k-1}\}$ and δ_{k-1} so that (u, δ_{k-1}) is a utility semi-representation of $\succsim$ restricted to $\{x_1, \dots, x_{k-1}\}$. Let $\kappa = \min\{u(x_i) - u(x_j) \colon i, j \in [k-1],\ x_i \succ x_j\}$. Observe that $\kappa > \delta_{k-1}$ and let $\delta_k \in (\delta_{k-1}, \kappa)$. Then, (u, δ_k) is a utility semi-representation of $\succsim$ restricted to $\{x_1, \dots, x_{k-1}\}$ and $|u(x_i) - u(x_j)| < \delta_k$ for all $i, j \in [k-1]$ with $x_i \sim x_j$ (note the strict inequality).

To extend u to x_k, we distinguish two cases. First, if $x_{k-1} \succ x_k$, let $u(x_k) < u(x_{k-1}) - \delta_k$. Second, if $x_{k-1} \sim x_k$, let $i = \min\{j \in [k-1] \colon x_j \sim x_k\}$ and $u(x_k) = u(x_i) - \delta_k$. Note that $x_j \sim x_i$ for all $j \in \{i, \dots, k-1\}$. To see this, observe that $x_i \hat{\succsim} x_j$ implies $x_i \succsim x_j$ by the last part of Lemma 3.12. Moreover, suppose for contradiction that $x_i \succ x_j$. Since $x_j \hat{\succsim} x_k$, we have $U(x_j) \subset U(x_k)$, and so $x_i \succ x_k$, which contradicts $x_i \sim x_k$. Similarly, we get that $x_j \sim x_k$. As (u, δ_k) represents $\succsim$ on $\{x_1, \dots, x_{k-1}\}$, it follows that $u(x_k) + \delta_k = u(x_i) \geq u(x_j) > u(x_i) - \delta_k = u(x_k)$ for all $j \in \{i, \dots, k-1\}$. For $j \in \{1, \dots, i-1\}$, we have $x_j \succ x_k$ by definition of i. By construction of u, $u(x_j) > u(x_i)$, and so $u(x_j) > u(x_i) = u(x_k) + \delta_k$. It follows that (u, δ_k) is a utility semi-representation of $\succsim$ restricted to $\{x_1, \dots, x_k\}$.

Having defined u on $X = \{x_1, \dots, x_n\}$ accordingly and letting $\delta = \delta_n$ gives that (u, δ) is a utility semi-representation of $\succsim$ on X. □

3.3 Exercises

Exercise 3.1 (Representation of countable sets). Let $X = \{x_1, x_2, \dots\}$ be countable and $\succsim$ be an anti-symmetric preference relation on X. Give an explicit utility representation u of $\succsim$, that is, a non-recursive definition of u in terms of $\succsim$.

Exercise 3.2 (Lexicographic preferences). Let $X = ([0,1] \cap \mathbb{Q}) \times [0,1]$ and define a relation $\succsim$ on X as follows. For all $(x_1, x_2), (y_1, y_2) \in X$,

$$(x_1, x_2) \succ (y_1, y_2) \text{ if and only if } x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 > y_2, \text{ and}$$
$$(x_1, x_2) \sim (y_1, y_2) \text{ if and only if } x_1 = y_1 \text{ and } x_2 = y_2.$$

(i) Prove that $\succsim$ is complete, transitive, and separable.

(ii) Give an explicit utility representation of $\succsim$.

Exercise 3.3 (Non-representable preferences). Give an example of a preference relation $\succsim$ on $[0,1]^2$ that does not admit a utility representation.

Exercise 3.4 (Composition of representations). Let $\succsim$ be a relation on a set X and $Y, Z \subset X$ so that $Y \cup Z = X$ and $y \succ z$ for all $y \in Y$ and $z \in Z$. Assume that the restrictions of $\succsim$ to Y and to Z both admit a utility representation. Show that $\succsim$ admits a utility representation.

Exercise 3.5 (Debreu's theorem). Prove Corollary 3.6 using Theorem 3.5. (*Hint:* You may use without proof the fact that every subset of $\mathbb{R}^d$ in the usual topology has a countable dense subset.)

Exercise 3.6 (Equivalent definitions of semi-orders). Prove Lemma 3.8.

Exercise 3.7 (Luce's theorem fails for infinitely many outcomes). Prove that the relation $\succsim$ on $X = [0,1] \cap \mathbb{Q}$ defined in Theorem 3.11 is a semi-order but does not admit a utility semi-representation.

4 Expected Utility

The preceding section answered the question of when preference relations admit a utility representation. When the set of outcomes X has additional structure, we can ask for utility representations that respect this structure. We encountered an example in Theorem 3.5, where X was a subset of $\mathbb{R}^d$ (or, more generally, a topological space), and the utility representation is required to be continuous.

This section examines the case when X has a convex structure. Utility representations that respect this structure are linear functions from X to $\mathbb{R}$. We start by considering the case when X is the set of lotteries over some set A of alternatives. The well-known expected utility theorem of von Neumann and Morgenstern (1947) characterizes the preference relations on lotteries that can be represented by expected utility representation. We then obtain a generalization of this result due to Herstein and Milnor (1953) when X is a convex subset of $\mathbb{R}^A$, where A is an arbitrary set. Lastly, we consider the case of lotteries over monetary prizes and study risk aversion.

4.1 Von Neumann-Morgenstern Expected Utility

Let A be a finite set of alternatives. We denote by $\mathcal{L}(A)$ the set of lotteries over A. That is, $\mathcal{L}(A) = \{p \in \mathbb{R}^A_+ \colon \sum_{a\in A} p(a) = 1\}$. By δ_a, we denote the lottery with probability 1 on $a \in A$. The lotteries δ_a are called degenerate lotteries. For $p, q \in \mathcal{L}(A)$ and $\alpha \in [0,1]$, we often write $p\alpha q = \alpha p + (1-\alpha)q$ for short for the convex combination of p and q with parameter α.[6]

Definition 4.1 (Expected utility representation)**.** Let $\succsim$ be a relation over $\mathcal{L}(A)$. A function $u\colon \mathcal{L}(A) \to \mathbb{R}$ is an expected utility representation of $\succsim$ if it represents $\succsim$ and is linear. In that case, we say that $\succsim$ admits an expected utility representation.

Linearity of u states that for all $p, q \in \mathcal{L}(A)$ and $\alpha \in [0,1]$,

$$u(p\alpha q) = \alpha u(p) + (1-\alpha)u(q).$$

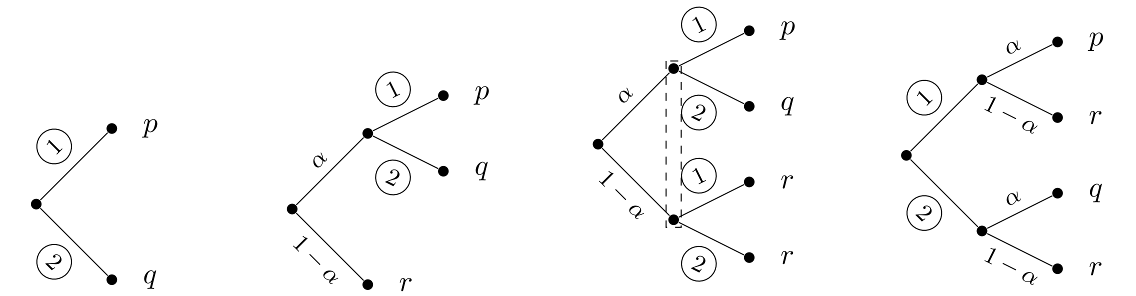
In particular, for all $p \in \mathcal{L}(A)$,

$$u(p) = \sum_{a\in A} p(a)u(\delta_a).$$

Hence, u is determined by its values on degenerate lotteries.

Economics abounds with cases where decision-makers are assumed to maximize expected utility. For example, much of game theory assumes that players evaluate mixed strategies by their expected utility (given their opponents' strategies). How, then, can we justify that decision-makers behave like expected utility maximizers (or rather should behave as such)? One of the seminal contributions of von Neumann and Morgenstern (1947) gives an answer in terms of three axioms on preferences over lotteries that are equivalent to utility maximization.

[6] Convex combinations of elements of $\mathbb{R}^A$ are defined component-wise. That is, the ith coordinate of $\alpha p+(1-\alpha)q$ is $\alpha p_i + (1-\alpha)q_i$.



(a) Decision (i) (b) Decision (ii) (c) Decision (iii) (d) Decision (iv)

Figure 4.1: The decisions (i) to (iv) depicted as extensive form games. The edges labeled with circled numbers denote decisions by the decision-maker. The edges with labels α and $1-\alpha$ are moves of Nature corresponding to the coin flips. The dashed rectangle in the game for Decision (iii) depicts the information set of the decision-maker after the move of Nature and corresponds to the fact that the decision-maker does not observe the outcome of the coin flip before her decision.

Definition 4.2 (Von Neumann-Morgenstern axioms)**.** A relation $\succsim$ on $\mathcal{L}(A) \subset \mathbb{R}^A$ satisfies the von Neumann-Morgenstern axioms if

$$\succsim \text{ is complete and transitive,} \tag{vNM1}$$

$$\text{for all } p, q, r \in \mathcal{L}(A) \text{ and } \alpha \in (0,1),\ p \succsim q \text{ if and only if } p\alpha r \succsim q\alpha r, \text{ and} \tag{vNM2}$$

$$\text{for all } p, q, r \in \mathcal{L}(A) \text{ with } p \succ q \succ r, \text{ there are } \alpha, \beta \in (0,1) \text{ with } p\alpha r \succ q \succ p\beta r. \tag{vNM3}$$

vNM1 requires that $\succsim$ is a preference relation and is known from the previous sections. vNM2 and vNM3 are known as independence and continuity.

Roughly speaking, independence requires that if p is preferred to q, then a coin flip between p and a third lottery r is preferred to a coin flip between q and r with the same coin used in both instances. To see why this makes sense, consider the following four choices (see also Gilboa, 2009, p. 82).

(i) You can choose either p or q.

(ii) A coin with probability α for heads is tossed. If it comes up heads, you can choose either p or q; otherwise, you get r.

(iii) Same as (ii), except that you have to commit to the choice between p and q before observing the outcome of the coin toss.

(iv) You can choose between two options. In the first, the same coin as above is tossed. If it comes up heads, you get p; otherwise you get r. The second option is the same except that p is replaced with q.

The four choices are depicted in Section 4.1 as extensive form games. Note that (iv) is the choice between $p\alpha r$ and $q\alpha r$. We relate (i) to (iv) by using (ii) and (iii) as intermediate steps.

When you are asked to act in (ii), you have the choice between p and q. At that point, r is a foregone possibility and should thus be irrelevant. Hence, your decision should be the same as in (i). The only difference between (ii) and (iii) is that you have to decide before knowing the outcome of the coin toss. Choosing the first option in (ii) and the second in (iii) or vice versa would mean that you are dynamically inconsistent. In (iii), you plan to make a particular choice, but when you see the outcome of the coin toss, you change your mind and act as in (ii). In other words, knowing if the coin has landed heads should not influence your choice. The difference between (iii) and (iv) is that in (iv), the coin is tossed after you make your choice. Equivalence between (iii) and (iv) means that it should not matter to your decision if the coin has been tossed, but you do not know the outcome, or the coin will be tossed after you make a choice. Here, it is relevant that your choice does not influence the coin toss (which seems like a fair assumption). In summary, independence requires that counterfactuals and the order of moves are irrelevant and that compound lotteries are reduced.

Like the separability axiom in Section 3, continuity (vNM3) cannot be empirically invalidated and has no apparent normative appeal. To see the former, suppose a decision-maker has exhibited the preferences $p \succ q \succ r$, and we want to determine if there is a value of $\alpha < 1$ so that she prefers $p\alpha r$ to q. Even if we try several values (close to 1) and find that none of them works, we cannot rule out that there is one (even closer to 1) that would work. However, one can test continuity with a thought experiment (which is chosen deliberately extreme). You are offered your favorite dish and can decide whether to eat it or not. The catch is that there is a small chance that it is poisoned and will kill you. Therefore, the three possible outcomes are enjoying the dish without any repercussions (p), refraining from eating it and staying hungry (q), and certain death (r). It seems fair to assume that you prefer the first outcome to the second and the second to the third. Thus, continuity would require you to decide to go for the dish if the probability of it being poisoned is small enough. This may seem absurd initially, but choices like this one are widespread (and unavoidable). The reason why deliberately accepting any chance of poisoning in the example may seem absurd is perhaps a failure to intuit very small numbers.

In contrast to the representation results in Section 3, the expected utility characterization comes with stronger uniqueness properties.

Definition 4.3 (Positive affine transformation)**.** Let $u, v\colon \mathcal{L}(A) \to \mathbb{R}$ be two expected utility representations. Then, v is a positive affine transformation of u if there are $a > 0$ and $b \in \mathbb{R}$ so that $v(p) = au(p) + b$ for all $p \in \mathcal{L}(A)$.

Expected utility maximization is one of the most common preference models in economics. It is thus hard to overestimate the importance of an axiomatic foundation for it.

Theorem 4.4 (von Neumann and Morgenstern, 1947)**.** *A relation $\succsim$ on $\mathcal{L}(A)$ satisfies vNM1–vNM3 if and only if it admits an expected utility representation $u\colon \mathcal{L}(A) \to \mathbb{R}$. Moreover, u is unique up to positive affine transformations.*

We omit the proof of Theorem 4.4 since it is a special case of a more general result we obtain in the next section.

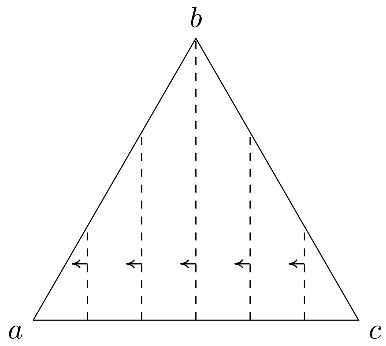


Figure 4.2: Illustration of expected utility preferences for $A = \{a, b, c\}$ and $u\colon A \to \mathbb{R}$ with $u(\delta_a) = 1$, $u(\delta_b) = 0$, and $u(\delta_c) = -1$. The indifference curves (dashed) are straight parallel lines. The arrows point in the direction of increasing preference.

It is often helpful to think about a preference relation in terms of its indifference curves, that is, equivalence classes of its symmetric part. Recall from Section 1 that for an equivalence relation $\sim$ and an outcome x, $[x]_\sim = \{y \in \mathcal{L}(A) : y \sim x\}$ is the equivalence class of x. The indifference curves of a preference relation $\succsim$ are thus the equivalence classes of $\sim$. If $\succsim$ admits an expected utility representation, its indifference curves are parallel hyperplanes of dimension at least $|A|-2$ (for example, lines if $|A| = 3$ and planes if $|A| = 4$). Figure 4.2 gives an example. To see that indifference curves are straight lines, suppose $p \sim q$ and $\hat{p}$ lies on the line through p and q. That is, $\hat{p} = p\alpha q$ for some $\alpha \in \mathbb{R}$. If $\alpha \in (0, 1)$, independence gives that

$$p \sim q \leftrightarrow p\alpha q \sim q\alpha q \leftrightarrow \hat{p} \sim q.$$

Hence, $\hat{p}$ lies on the same indifference curve as p and q. The argument is similar for the remaining values of α. To see that they are parallel, assume $p \sim q$ and suppose $\hat{p}$ and $\hat{q}$ are lotteries so that the line through p and q is parallel to the line through $\hat{p}$ and $\hat{q}$. Formally, this means that $\hat{p} - \hat{q}$ is a multiple of $p - q$, and we assume for convenience that both differences are equal. Independence implies that

$$p \sim q \leftrightarrow \frac{1}{2}\,p + \frac{1}{2}\left(\frac{1}{2}\,\hat{p} + \frac{1}{2}\,\hat{q}\right) \sim \frac{1}{2}\,q + \frac{1}{2}\left(\frac{1}{2}\,\hat{p} + \frac{1}{2}\,\hat{q}\right).$$

Then, using $\frac{1}{4}\,p = \frac{1}{4}\,(q + \hat{p} - \hat{q})$ gives

$$\frac{1}{2}\,p + \frac{1}{2}\left(\frac{1}{2}\,\hat{p} + \frac{1}{2}\,\hat{q}\right) = \frac{1}{2}\,\hat{p} + \frac{1}{2}\left(\frac{1}{2}\,p + \frac{1}{2}\,q\right).$$

Similarly,

$$\frac{1}{2}\,q + \frac{1}{2}\left(\frac{1}{2}\,\hat{p} + \frac{1}{2}\,\hat{q}\right) = \frac{1}{2}\,\hat{q} + \frac{1}{2}\left(\frac{1}{2}\,p + \frac{1}{2}\,q\right).$$

Another application of independence gives $\hat{p} \sim \hat{q}$ as desired.

Experimental economists have examined in great depth the extent to which real-world decision-makers behave like expected utility maximizers. In that pursuit, they conceived experiments in which one of the axioms is frequently violated. One of the earliest and most well-known ones is due to Allais (1953) and is known as the Allais paradox.

Consider the following four lotteries over monetary prizes.

$$\$1 \text{ million}\colon 100\% \tag{p_1}$$

$$\$1 \text{ million}\colon 90\%;\ \$0\colon 1\%;\ \$5 \text{ million}\colon 9\% \tag{q_1}$$

$$\$0\colon 90\%;\ \$1 \text{ million}\colon 10\% \tag{p_2}$$

$$\$0\colon 91\%;\ \$5 \text{ million}\colon 9\% \tag{q_2}$$

Participants are asked to make two fictitious choices: first, between p_1 and q_1, and second, between p_2 and q_2. A significant fraction of participants chooses p_1 in the first situation and q_2 in the second, so that $p_1 \succ q_1$ and $q_2 \succ p_2$. Allais presented this finding as a violation of the independence axiom. To see this, consider four auxiliary lotteries.

$$\$1 \text{ million}\colon 100\% \tag{p}$$

$$\$0\colon 10\%;\ \$5 \text{ million}\colon 90\% \tag{q}$$

$$\$1 \text{ million}\colon 100\% \tag{r_1}$$

$$\$0\colon 100\% \tag{r_2}$$

Then, we have $p_1 = .1\,p + .9\,r_1$ and $q_1 = .1\,q + .9\,r_1$. So since $p_1 \succ q_1$, independence would require that $p \succ q$. On other hand, $p_2 = .1\,p + .9\,r_2$ and $q_2 = .1\,q + .9\,r_2$. Since $q_2 \succ p_2$, independence would now require $q \succ p$. Since this is a contradiction, the preferences violate independence.

A common explanation of the Allais paradox is the certainty effect popularized by Kahneman and Tversky (1979). It asserts that decision-makers prefer certain outcomes to risky ones. Hence the preference for p_1 over q_1. On the other hand, both p_2 and q_2 involve risk, so the higher expected payoff of q_2 is decisive.

Many other experiments have revealed violations of independence as well as transitivity. As a reaction, decision theorists have considered weakenings of the von Neumann-Morgenstern axioms and derived alternative theories of decision-making under risk. Some notable examples are weighted linear utility theory (Chew, 1983), rank-dependent utility theory (Quiggin, 1993), regret theory (Loomes and Sugden, 1982), prospect theory (Kahneman and Tversky, 1979), disappointment aversion (Gul, 1991), and skew-symmetric bilinear utility theory (Fishburn, 1982).

4.2 The Herstein-Milnor Theorem

This section aims to prove a more general version of Theorem 4.4. We will use this theorem not only to establish the latter result but also several times in the subsequent sections. We start by introducing some terminology.

Let A be an arbitrary set. A subset X of $\mathbb{R}^A$ is convex if for all $p, q \in X$ and $\alpha \in [0,1]$, $\alpha p + (1-\alpha)q \in X$. As before, we sometimes write $p\alpha q$ for short. Note that $p\alpha q = q(1-\alpha)p$. If X is convex, a function $u\colon X \to \mathbb{R}$ is linear if for all $p, q \in X$ and $\alpha \in [0,1]$,

$$u(p\alpha q) = \alpha u(p) + (1-\alpha)u(q).$$

We say $v\colon X \to \mathbb{R}$ is a positive affine transformation of u if there are $a > 0$ and $b \in \mathbb{R}$ so that $v(p) = au(p) + b$ for all $p \in X$.

Definition 4.5 (Linear utility representation)**.** Let X be convex and $\succsim$ be a relation on X. A function $u\colon X \to \mathbb{R}$ is a linear utility representation of $\succsim$ if it represents $\succsim$ and is linear. In that case, we say that $\succsim$ admits a linear utility representation.

Note that $\mathcal{L}(A)$ is convex, and if u is an expected utility representation of $\succsim$, then it is linear. Hence, expected utility representations are linear utility representations.

The axioms vNM1–vNM3 introduced in Section 4.1 carry over verbatim to the case when X is an arbitrary convex subset of $\mathbb{R}^A$ instead of $\mathcal{L}(A)$.

Theorem 4.6 (Mixture space theorem, Herstein and Milnor, 1953)**.** *Let $X \subset \mathbb{R}^A$ be convex. A relation $\succsim$ on X satisfies vNM1, vNM2, and vNM3 if and only if there is a linear function $u\colon X \to \mathbb{R}$ representing $\succsim$. Moreover, u is unique up to positive affine transformations.*

The proof uses four lemmas. Throughout, X is a convex subset of $\mathbb{R}^A$.

Lemma 4.7. *Let $\succsim$ be a relation on X satisfying vNM1–vNM3. Then, for all $p, q, r, s \in X$ and $\alpha, \beta \in [0,1]$,*

$$p \succ q,\ \alpha > \beta \text{ implies } p\alpha q \succ p\beta q, \tag{1}$$

$$p \sim q \text{ implies } p \sim p\alpha q, \text{ and} \tag{2}$$

$$p \succsim q,\ r \succsim s \text{ implies } p\alpha r \succsim q\alpha s. \tag{3}$$

Moreover, if one of the relations in the antecedent of (3) is strict, so is the relation in the consequent.

Proof. (1) Applying vNM2 once with $r = p$ and once with $r = q$, we get $p = p(1-\alpha)p \succ q(1-\alpha)p = p\alpha q$ and $p\alpha q \succ q\alpha q = q$. Thus, $p \succ p\alpha q \succ q$. Let $\gamma = \frac{\beta}{\alpha} \in [0,1)$. Using the preceding conclusion, it follows that $p\alpha q \succ (p\alpha q)\gamma q = p(\alpha\gamma)q = p\beta q$.

(2) Applying independence with $r = p$ and using $p \sim q$ gives $p = p(1-\alpha)p \sim q(1-\alpha)p = p\alpha q$.

(3) The cases $\alpha = 0$ and $\alpha = 1$ are trivial. So we assume $\alpha \in (0,1)$. Since $p \succsim q$ and $r \succsim s$, independence gives

$$p\alpha r \succsim q\alpha r \text{ and } q\alpha r = r(1-\alpha)q \succsim s(1-\alpha)q = q\alpha s.$$

Transitivity of $\succsim$ then implies $p\alpha r \succsim q\alpha s$. Moreover, transitivity implies that if $p \succ q$ or $r \succ s$, then $p\alpha r \succ q\alpha s$. □

Lemma 4.8. *Let $\succsim$ be a relation on X satisfying vNM1–vNM3. Then, for all $p, q, r \in X$ with $p \succ q \succ r$, the sets*

$$A^+ = \{\alpha \in [0,1] \colon p\alpha r \succ q\} \text{ and } A^- = \{\alpha \in [0,1] \colon q \succ p\alpha r\}$$

are non-empty, convex, and open in $[0,1]$.

Proof. By vNM3, A^+ is non-empty. To prove that A^+ is convex, let $\alpha \in A^+$ and let $\beta \in [\alpha, 1]$. We prove that $\beta \in A^+$. Note that $1 \in A^+$. vNM2 applied to the triple $p\alpha r$, q, and p gives

$$p\beta r = (p\alpha r)(\frac{1-\beta}{1-\alpha})p \succ q(\frac{1-\beta}{1-\alpha})p.$$

Moreover, by Lemma 4.7, $q(\frac{1-\beta}{1-\alpha})p \succsim q$. Since by vNM1, $\succsim$ is transitive it follows that $p\beta q \succ q$. Hence, $\beta \in A^+$. Since $\beta \in [\alpha, 1]$ was arbitrary, A^+ is convex. Similarly, A^- is convex.

We prove that A^+ is open. Convexity of A^+ implies that either $A^+ = (\alpha^*, 1]$ or $A^+ = [\alpha^*, 1]$ for some $\alpha^* \in (0, 1)$. In the first case, A^+ is open in $[0, 1]$. So assume for contradiction that $A^+ = [\alpha^*, 1]$. Then, $p\alpha^* r \succ q \succ r$. So vNM3 implies that $(p\alpha^* r)\alpha r \succ q$ for some $\alpha \in (0, 1)$. But $(p\alpha^* r)\alpha r = \alpha(\alpha^* p + (1-\alpha^*)r) + (1-\alpha)r = \alpha\alpha^* p + (1-\alpha\alpha^*)r$, so that $\alpha\alpha^* \in A^+$. But this is a contradiction since $\alpha\alpha^* < \alpha^*$. Similarly, one shows that A^- is empty or open in $[0, 1]$. □

Lemma 4.9. *Let $\succsim$ be a relation on X satisfying vNM1–vNM3. Then, for all $p, q, r \in X$ with $p \succ q \succ r$, there is a unique $\alpha^* \in (0, 1)$ so that $p\alpha^* r \sim q$.*

Proof. Let

$$A^+ = \{\alpha \in [0, 1] \colon p\alpha r \succ q\} \text{ and } A^- = \{\alpha \in [0, 1] \colon q \succ p\alpha r\},$$

and $\alpha^+ = \inf A^+$ and $\alpha^- = \sup A^-$. By Lemma 4.8, A^+ and A^- are non-empty, convex, and open. Non-emptiness implies that α^+ and α^- are well-defined and convexity that $\alpha^+ \geq \alpha^-$. If $\alpha^+ > \alpha^-$, let $\alpha^+ > \alpha > \beta > \alpha^-$ and observe that $p\alpha r \sim q \sim p\beta r$ by definition of α^+ and α^-. On the other hand, Lemma 4.7(1) gives $p\alpha r \succ p\beta r$, which is a contradiction. Hence, $\alpha^+ = \alpha^-$. Letting $\alpha^* = \alpha^+$ and using that A^+ and A^- are open gives $p\alpha^* r \sim q$. □

The last lemma is needed for the uniqueness statement in Theorem 4.6.

Lemma 4.10. *Let $\succsim$ be a relation on X satisfying vNM1–vNM3. If u, v are linear functions representing $\succsim$, then there are $a > 0$ and $b \in \mathbb{R}$ so that $v(p) = au(p) + b$ for all $p \in X$.*

Proof. If $p \sim q$ for all $p, q \in X$, u and v are both constant, and the statement is trivial. Otherwise, let $\overline{p}, \underline{p} \in X$ with $\overline{p} \succ \underline{p}$. Replacing u by $\tilde{u}$ with $\tilde{u}(p) = \frac{u(p)-u(\underline{p})}{u(\overline{p})-u(\underline{p})}$, it is without loss to assume that $u(\overline{p}) = 1$ and $u(\underline{p}) = 0$. Let $a = v(\overline{p}) - v(\underline{p})$ and $b = v(\underline{p})$.

Let $p \in X$ and $\alpha = u(p)$. There are three cases. If $\alpha \in [0, 1]$, we get $p \sim \alpha\overline{p} + (1-\alpha)\underline{p}$. Hence,

$$v(p) = \alpha v(\overline{p}) + (1-\alpha)v(\underline{p}) = \alpha(v(\overline{p}) - v(\underline{p})) + v(\underline{p}) = a\alpha + b = au(p) + b.$$

If $\alpha > 1$, we get $\overline{p} \sim \frac{1}{\alpha}p + (1 - \frac{1}{\alpha})\underline{p}$. Applying v and rearranging yields

$$v(p) = \alpha v(\overline{p}) - (\alpha - 1)v(\underline{p}) = \alpha(v(\overline{p}) - v(\underline{p})) + v(\underline{p}) = au(p) + b.$$

If $\alpha < 0$, we get $\underline{p} \sim \frac{-\alpha}{1-\alpha}\overline{p} + \frac{1}{1-\alpha}p$. Hence,

$$v(p) = \alpha v(\overline{p}) + (1-\alpha)v(\underline{p}) = au(p) + b.$$

Hence, $v(p) = au(p) + b$ for all $p \in X$. □

Proof of Theorem 4.6. If $p \sim q$ for all $p, q \in X$, any constant function represents $\succsim$, and constant functions are linear. Assume now that there are $\overline{p}$ and $\underline{p}$ in X with $\overline{p} \succ \underline{p}$. For any $p \in X$, let

$$u(p) = \begin{cases} \alpha & \text{if } \overline{p} \succsim p \succsim \underline{p} \text{ and } p \sim \overline{p}\alpha\underline{p}, \\ \frac{1}{\alpha} & \text{if } p \succ \overline{p} \text{ and } p\alpha\underline{p} \sim \overline{p}, \text{ and} \\ -\frac{\alpha}{1-\alpha} & \text{if } \underline{p} \succ p \text{ and } \overline{p}\alpha p \sim \underline{p}. \end{cases}$$

Lemma 4.9 shows that u is well-defined, that is, uniquely defined for all p. Note that $u(p) > 1$ if $p \succ \overline{p}$, $u(p) \in [0, 1]$ if $\overline{p} \succsim p \succsim \underline{p}$, and $u(p) < 0$ if $\underline{p} \succ p$.

Showing that u is linear requires checking six cases. For instance, let $p, q \in X$ with $\overline{p} \succsim p, q \succsim \underline{p}$ and $\beta \in [0, 1]$. Then, $\overline{p} \succsim p\beta q \succsim \underline{p}$ by Lemma 4.7(1). Assume $u(p) = \alpha_p$ and $u(q) = \alpha_q$, where $p \sim \overline{p}\alpha_p\underline{p}$ and $q \sim \overline{p}\alpha_q\underline{p}$. Lemma 4.7(3) implies

$$p\beta q \sim (\overline{p}\alpha_p\underline{p})\beta(\overline{p}\alpha_q\underline{p}) = \overline{p}(\beta\alpha_p + (1-\beta)\alpha_q)\underline{p}.$$

Hence,

$$u(p\beta q) = \beta\alpha_p + (1-\beta)\alpha_q = \beta u(p) + (1-\beta)u(q),$$

which proves linearity in this case. The remaining five cases are similar.

To see that u represents $\succsim$, let $p, q \in X$. If $p \sim q$, then clearly $u(p) = u(q)$. Now let $p \succ q$. If $p \succsim \underline{p}$ and $\underline{p} \succ q$, then $u(p) \geq 0 > u(q)$, and if $p \succ \underline{p}$ and $\underline{p} \succsim q$, then $u(p) > 0 \geq u(q)$. If $q \succ \underline{p}$, let $\alpha \in (0, 1)$ so that $p\alpha\underline{p} \sim q$. Then,

$$u(p) > \alpha u(p) + (1-\alpha)0 = \alpha u(p) + (1-\alpha)u(\underline{p}) = u(p\alpha\underline{p}) = u(q).$$

The case $\underline{p} \succ p$ is similar.

The uniqueness of u up to positive affine transformations follows from Lemma 4.10. □

Remark 4.11 (Generalizations of Theorem 4.6). The original result of Herstein and Milnor (1953) generalizes Theorem 4.6 in two respects. First, it uses the following weaker notion of the independence axiom.

$$\text{For all } p, q, r \in X,\ p \succ q \text{ implies } \frac{1}{2}\,p + \frac{1}{2}\,r \succ \frac{1}{2}\,q + \frac{1}{2}\,r. \tag{vNM2$^-$}$$

Second, they prove it for arbitrary mixture spaces. A mixture allows taking convex combinations of its elements, and this operation possesses most but not all of the properties familiar from $\mathbb{R}^A$.

4.3 Lotteries Over Monetary Prizes and Risk Aversion

A particular case of the setting in Section 4.1 is when the alternatives are monetary prizes, so $A = \mathbb{R}$. Outcomes are thus lotteries over monetary prizes. This case is special for two reasons. First, assuming that decision-makers prefer higher prizes to lower prizes, it comes with a natural order over alternatives. Adding this assumption to those in Theorem 4.4 allows us to obtain an expected utility representation based on a strictly increasing utility function $\bar{u}\colon A \to \mathbb{R}$. Second, using the additive structure of $\mathbb{R}$, one can define the expected payoff of a lottery and, with it, a notion of risk aversion: every lottery is less preferred than the (degenerate) lottery that pays the expected payoff of the former for sure. This will imply that $\bar{u}$ is concave so that the marginal utility of money is decreasing.

To motivate the proceeding concepts, let us consider the so-called St. Petersburg paradox. A fair coin is tossed until it comes up heads for the first time. If the game stops after k tosses, you are paid \$$2^k$. This results in the following lottery.

$$\$2\colon \frac{1}{2};\ \$4\colon \frac{1}{4};\ \$8\colon \frac{1}{8};\ \dots\ \$2^k\colon \frac{1}{2^k};\ \dots$$

The expected payoff of this lottery is

$$\$2 \cdot \frac{1}{2} + \$4 \cdot \frac{1}{4} + \$8 \cdot \frac{1}{8} + \cdots = 1 + 1 + 1 + \cdots = \infty.$$

So, if you are maximizing your expected payoff, you should be willing to pay any amount of money to participate in the game. Since, empirically, most people would not do this, expected payoff maximization does not seem to be a common objective in this situation. Bernoulli (1738) suggested that people maximize expected utility rather than payoff. If the utility increases sufficiently slowly in money, then the expected utility of the game is finite. For example, suppose the utility of \$$n$ is $\log_2 n$. Then, the game has expected utility

$$1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \cdots = 2.$$

This specific utility function is concave, a property closely related to risk-aversion as we will see.

Let $A = \mathbb{R}$ and denote by $\mathcal{L}(A)$ the set of simple lotteries on A. That is, $\mathcal{L}(A) = \{p \in \mathbb{R}^A_+ \colon p(a) = 0 \text{ for all but finitely many } a \in A \text{ and } \sum_{a \in A} p(a) = 1\}$. This extends our previous definition of $\mathcal{L}(A)$ to infinite A. The expected payoff of a lottery $p \in \mathcal{L}(A)$ is

$$\mathrm{E}(p) = \sum_{a \in A} p(a)a.$$

Note that $\mathrm{E}(p)$ is well-defined since p is simple. The support of p is the set of alternatives to which p assigns positive probability. That is, $\mathrm{supp}(p) = \{a \in A \colon p(a) > 0\}$.

Definition 4.12 (Monotonic expected utility representation)**.** A relation $\succsim$ on $\mathcal{L}(A)$ admits a monotonic expected utility representation if there is a strictly increasing function $\bar{u}\colon A \to \mathbb{R}$ so that $\succsim$ is represented by

$$u(p) = \sum_{a \in A} p(a)\bar{u}(a).$$

Note that any u as above is also an expected utility representation in the sense of Definition 4.1 since it is linear. Our characterization of preferences $\succsim$ over $\mathcal{L}(A)$ that admit a monotonic expected utility representation will be based on the von Neumann-Morgenstern axioms vNM1–vNM3 plus a new one called monotonicity. It requires that higher sure prizes are preferred to lower sure prizes. More precisely, a relation $\succsim$ on $\mathcal{L}(A)$ is monotonic if

$$\delta_a \succ \delta_b \text{ for all } a > b. \tag{MON}$$

Theorem 4.13 (von Neumann-Morgenstern theorem on monetary prizes)**.** *A relation $\succsim$ on $\mathcal{L}(A)$ satisfies vNM1, vNM2, vNM3, and MON if and only if it admits a monotonic expected utility representation.*

Proof. We prove the "only if" part. By Theorem 4.6, there is a linear function $u\colon \mathcal{L}(A) \to \mathbb{R}$ representing $\succsim$. Define $\bar{u}\colon A \to \mathbb{R}$ by letting $\bar{u}(a) = u(\delta_a)$. Since u is linear, for all $p \in \mathcal{L}(A)$,

$$u(p) = \sum_{a \in A} p(a)u(\delta_a) = \sum_{a \in A} p(a)\bar{u}(a).$$

Lastly, for $a, b \in A$ with $a > b$, MON implies

$$\bar{u}(a) = u(\delta_a) > u(\delta_b) = \bar{u}(b).$$

It follows that $\bar{u}$ is strictly increasing.

To prove the "if" part, assume that $\succsim$ admits a monotonic expected utility representation (in the sense of Definition 4.12). By Theorem 4.6, $\succsim$ satisfies vNM1, vNM2, and vNM3. To check monotonicity, let $a, b \in A$ with $a > b$. Then,

$$u(\delta_a) = \bar{u}(a) > \bar{u}(b) = u(\delta_b),$$

where the inequality uses that $\bar{u}$ is strictly increasing. Since u represents $\succsim$, it follows that $\delta_a \succ \delta_b$. $\square$

The certainty equivalent of a lottery is the amount of money a decision-maker would be willing to pay in exchange for the lottery.

Definition 4.14 (Certainty equivalent)**.** Let $\succsim$ be a preference relation on $\mathcal{L}(A)$ satisfying MON and $p \in \mathcal{L}(A)$. Then, $a \in A$ is the certainty equivalent of p if $p \sim \delta_a$. In that case, we write $C(p) = a$.

Note that the monotonicity of $\succsim$ ensures that the certainty equivalent is unique whenever it exists. If $\succsim$ admits a monotonic expected utility representation u, then the certainty equivalent of p (if it exists) is $\bar{u}^{-1}(u(p))$. By the intermediate value theorem, certainty equivalents exist if $\bar{u}$ is continuous.

Example 4.15 (Certainty equivalent)**.** Let $u\colon \mathcal{L}(A) \to \mathbb{R}$ be the linear utility function induced by $\bar{u}(a) = 2a$ if $a \leq 0$ and $u(a) = a$ if $a \geq 0$. Let p be the lottery that pays 1 and -1 with probability $\frac{1}{2}$ each. Then, $u(p) = \frac{1}{2}\bar{u}(1) + \frac{1}{2}\bar{u}(-1) = -\frac{1}{2} = \bar{u}(-\frac{1}{4})$. Hence, $C(p) = -\frac{1}{4}$. Note that $-\frac{1}{4} < 0 = E(p)$.

Roughly speaking, risk aversion means that the certainty equivalent of a lottery is lower than its expected payoff. In other words, the decision-maker would rather take the expected payoff for sure than the lottery. We will see several equivalent definitions soon.

Definition 4.16 (Risk aversion)**.** A preference relation $\succsim$ on $\mathcal{L}(A)$ satisfying MON is risk-averse if for all $p \in \mathcal{L}(A)$, $C(p)$ exists and $\mathrm{E}(p) \geq C(p)$.

Risk aversion is closely tied to second-order stochastic dominance. Intuitively, one lottery second-order stochastically dominates another if the former is less spread out and has no less expected payoff than the latter. The formal definition uses the cumulative distribution function $F_p \colon A \to \mathbb{R}$ of a lottery p defined by

$$F_p(a) = \sum_{x \leq a} p(x).$$

Definition 4.17 (Second-order stochastic dominance (SOSD))**.** Let $p, q \in \mathcal{L}(A)$. Then, p second-order stochastically dominates q if for all $a \in A$,

$$\int_{-\infty}^{a} F_p(x) - F_q(x) dx \leq 0.$$

The dominance is strict if the inequality is strict for at least one $a \in A$.

One can show that $\mathrm{E}(p) = \int_0^\infty (1 - F_p(x))dx - \int_{-\infty}^0 F_p(x)dx$. Hence, if p second-order stochastically dominates q, then $\mathrm{E}(p) \geq \mathrm{E}(q)$.

Example 4.18 (Second-order stochastic dominance)**.** Let $p = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ and $q = \frac{1}{3}\delta_{-2} + \frac{1}{3}\delta_0 + \frac{1}{3}\delta_2$. Note that both p and q have an expected payoff of 0. We find that

$$F_p(a) = \begin{cases} 0 & \text{if } a < -1, \\ \frac{1}{2} & \text{if } -1 \leq a < 1, \text{ and} \\ 1 & \text{if } a \geq 1. \end{cases} \qquad F_q(a) = \begin{cases} 0 & \text{if } a < -2, \\ \frac{1}{3} & \text{if } -2 \leq a < 0, \\ \frac{2}{3} & \text{if } 0 \leq a < 2, \text{ and} \\ 1 & \text{if } a \geq 2. \end{cases}$$

Thus,

$$\int_{-\infty}^{a} F_p(x)dx = \begin{cases} 0 & \text{if } a \leq -1, \\ \frac{1}{2}(a+1) & \text{if } -1 \leq a \leq 1, \text{ and} \\ a & \text{if } a \geq 1. \end{cases} \quad \int_{-\infty}^{a} F_q(x)dx = \begin{cases} 0 & \text{if } a \leq -2, \\ \frac{1}{3}(a+2) & \text{if } -2 \leq a \leq 0, \\ \frac{2}{3}(a+1) & \text{if } 0 \leq a \leq 2, \text{ and} \\ a & \text{if } a \geq 2. \end{cases}$$

One can then check that p second-order stochastically dominates q.

Recall that a function $\bar{u} \colon \mathbb{R} \to \mathbb{R}$ is concave if $\bar{u}(\alpha a + (1-\alpha)b) \geq \alpha\bar{u}(a) + (1-\alpha)\bar{u}(b)$. It turns out that one lottery second-order stochastically dominates another if and only if any expected utility-maximizing decision-maker with a concave utility function prefers the former to the latter.

Proposition 4.19 (Equivalent notions of second-order stochastic dominance)**.** *Let $p, q \in \mathcal{L}(A)$. Then, the following are equivalent.*

(i) p second-order stochastically dominates q.

(ii) For every non-decreasing and concave function $\bar{u}\colon A \to \mathbb{R}$,

$$\sum_{a\in A} p(a)\bar{u}(a) \geq \sum_{a\in A} q(a)\bar{u}(a).$$

Proof. (ii) $\to$ (i) For $a \in A$, define $\bar{u}_a\colon A \to \mathbb{R}$ by letting $\bar{u}_a(x) = x - a$ for $x \leq a$ and $\bar{u}_a(x) = 0$ for $x \geq a$. Note that $\bar{u}_a$ is non-decreasing and concave. Denote by $\mathbf{1}_{(-\infty,x]}$ the indicator function of the interval $(-\infty, x]$. For $a \in A$, we have

$$\int_{-\infty}^{a} F_p(x)dx = \int_{-\infty}^{a} \sum_{y\leq x} p(y)dx = \int_{-\infty}^{a} \sum_{y\in A} \mathbf{1}_{(-\infty,x]}(y)p(y)dx = \sum_{y\in A} p(y) \int_{-\infty}^{a} \mathbf{1}_{(-\infty,x]}(y)dx$$
$$= \sum_{y\in A} p(y) \int_{\min\{y,a\}}^{a} 1dx = -\sum_{y\in A} p(y)\bar{u}_a(y).$$

By assumption, we thus have

$$\int_{-\infty}^{a} F_p(x)dx = -\sum_{y\in A} p(y)\bar{u}_a(y) \leq -\sum_{y\in A} q(y)\bar{u}_a(y) = \int_{-\infty}^{a} F_q(x)dx,$$

which proves the claim.

(i) $\to$ (ii) By the preceding argument, we have that for all $a \in A$,

$$\sum_{x\in A} p(x)\bar{u}_a(x) \geq \sum_{x\in A} q(x)\bar{u}_a(x).$$

Hence, the same inequality holds for all $\bar{u}$ of the form $\bar{u} = \sum_{i=1}^{n} \alpha_i \bar{u}_{a_i} + b$ for some $\alpha_i > 0$, $b \in \mathbb{R}$, and $a_i \in A$. Since any non-decreasing and concave function can be approximated uniformly by such $\bar{u}$ on the support of p and q, the inequality holds for any non-decreasing and concave function.

□

The main result of this section gives several equivalent formulations of risk aversion.

Theorem 4.20. *Let $\succsim$ be a relation on $\mathcal{L}(A)$ that admits a monotonic expected utility representation given by $\bar{u}\colon A \to \mathbb{R}$. Then, the following are equivalent.*

(i) $\succsim$ is risk-averse.

(ii) $\delta_{\frac{1}{2}a+\frac{1}{2}b} \succsim \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$ for all $a, b \in A$.

(iii) $\bar{u}$ is concave.

(iv) $p \succsim q$ whenever p second-order stochastically dominates q.

Proof. (i) $\to$ (ii) Let $a, b \in A$ with $a \geq b$ and $p = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$. Note that $\mathrm{E}(p) = \frac{1}{2}a + \frac{1}{2}b$. Risk aversion implies that $E(p) \geq C(p)$, and so since $\bar{u}$ is strictly increasing,

$$\delta_{\frac{1}{2}a+\frac{1}{2}b} = \delta_{E(p)} \succsim \delta_{C(p)} \sim p.$$

(ii) $\to$ (iii) By assumption, $u(p) = \sum_{a \in A} p(a)\bar{u}(a)$ represents $\succsim$. So for $a, b \in A$,

$$\bar{u}(\frac{1}{2}a + \frac{1}{2}b) = u(\delta_{\frac{1}{2}a+\frac{1}{2}b}) \geq u(\frac{1}{2}\delta_a + \frac{1}{2}\delta_b) = \frac{1}{2}\bar{u}(a) + \frac{1}{2}\bar{u}(b).$$

Applying this to a and $\frac{1}{2}a + \frac{1}{2}b$, and to $\frac{1}{2}a + \frac{1}{2}b$ and b, and using the preceding inequality, we get

$$\bar{u}(\frac{3}{4}a + \frac{1}{4}b) \geq \frac{3}{4}\bar{u}(a) + \frac{1}{4}\bar{u}(b) \quad \text{and} \quad \bar{u}(\frac{1}{4}a + \frac{3}{4}b) \geq \frac{1}{4}\bar{u}(a) + \frac{3}{4}\bar{u}(b).$$

Repeating this argument, we get that $\bar{u}(\alpha a + (1-\alpha)b) \geq \alpha\bar{u}(a) + (1-\alpha)\bar{u}(b)$ for all $\alpha \in \mathbb{D} = \{\frac{k}{2^n} : n \in \mathbb{N},\ k \in [2^n]\}$. Now if $\alpha \in [0,1]$ is arbitrary, we get since $a \geq b$ and $\bar{u}$ is strictly increasing that

$$\begin{aligned}\bar{u}(\alpha a + (1-\alpha)b) &\geq \sup_{\beta \in \mathbb{D},\ \beta \leq \alpha} \bar{u}(\beta a + (1-\beta)b) \geq \sup_{\beta \in \mathbb{D},\ \beta \leq \alpha} \beta\bar{u}(a) + (1-\beta)\bar{u}(b) \\ &= \alpha\bar{u}(a) + (1-\alpha)\bar{u}(b).\end{aligned}$$

Hence, $\bar{u}$ is concave.

(iii) $\to$ (iv) Assume that p second-order stochastically dominates q. By Proposition 4.19, for every non-decreasing and concave function $v : A \to \mathbb{R}$,

$$\sum_{a \in A} p(a)v(a) \geq \sum_{a \in A} q(a)v(a).$$

In particular, this inequality holds for $v = \bar{u}$. Since $\bar{u}$ represents $\succsim$, it follows that $p \succsim q$.

(iv) $\to$ (ii) Let $a, b \in A$ and observe that $\delta_{\frac{1}{2}a+\frac{1}{2}b}$ second-order stochastically dominates $\frac{1}{2}\delta_a + \frac{1}{2}\delta_b$. Hence, $\delta_{\frac{1}{2}a+\frac{1}{2}b} \succsim \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$.

(iii) $\to$ (i) Let $p \in \mathcal{L}(A)$. Since $\bar{u}$ is concave, Jensen's inequality implies that

$$u(\delta_{\mathrm{E}(p)}) = \bar{u}(\mathrm{E}(p)) = \bar{u}(\sum_{a \in A} p(a)a) \geq \sum_{a \in A} p(a)\bar{u}(a) = u(p).$$

Hence, $\delta_{\mathrm{E}(p)} \succsim p \sim \delta_{C(p)}$ and so $\mathrm{E}(p) \geq C(p)$ since $\bar{u}$ is strictly increasing. □

Combining Theorem 4.13 and Theorem 4.20 characterizes risk-averse expected utility preferences.

Corollary 4.21 (von Neumann-Morgenstern theorem on monetary prizes with risk-aversion). *A relation $\succsim$ on $\mathcal{L}(A)$ satisfies vNM1, vNM2, vNM3, MON, and risk-aversion if and only if it admits a monotonic expected utility representation given by a concave function $\bar{u}$.*

4.4 Exercises

Exercise 4.1 (Independence of the vNM axioms). Show that all three of vNM1, vNM2, and vNM3 are needed in Theorem 4.4. That is, show that for any two of the axioms, there is a relation that satisfies these two axioms but violates the third.

Exercise 4.2 (vNM Theorem for two alternatives). Let $A = \{a, b\}$ be some two-element set and $\succsim$ a complete relation on $\mathcal{L}(A)$ that satisfies vNM2. Prove that there is a linear function $u \colon \mathcal{L}(A) \to \mathbb{R}$ representing $\succsim$. (Note the absence of vNM1 and vNM3.)

Exercise 4.3 (Weighted linear utility). Let A be a finite set and $\succsim$ a relation on $\mathcal{L}(A)$. A function $uw \colon \mathcal{L}(A) \to \mathbb{R}$ is a weighed linear utility representation of $\succsim$ if it represents $\succsim$ and there are $\bar{u} \colon A \to \mathbb{R}$ and $\bar{w} \colon A \to \mathbb{R}_{++}$ so that for all $p \in \mathcal{L}(A)$

$$uw(p) = \frac{\sum_{a \in A} p(a)\bar{u}(a)}{\sum_{a \in A} p(a)\bar{w}(a)}.$$

(i) Let $A = \{a, b, c\}$ and define $\bar{u}$ and $\bar{w}$ by letting $\bar{u}(a) = 1$, $\bar{u}(b) = 0$, and $\bar{u}(c) = -1$, and $\bar{w}(a) = 2$, $\bar{w}(b) = 1$, and $\bar{w}(c) = 1$. Sketch the indifference curves of the weighted linear utility preferences induced by $\bar{u}$ and $\bar{w}$.

(ii) Prove that every relation with a linear utility representation admits a weighted linear utility representation.

(iii) Let $\succsim$ be a relation on $\mathcal{L}(A)$ that admits a weighed linear utility representation. Prove that $\succsim$ satisfies vNM1 and vNM3 but not in general vNM2.

Exercise 4.4 (First-oder stochastic dominance). Let $A = \mathbb{R}$. For $p, q \in \mathcal{L}(A)$, p first-oder stochastically dominates (FOSD) q if $F_p(a) - F_q(a) \leq 0$ for all $a \in A$.

(i) Show that p first-oder stochastically dominates q if and only if for every non-decreasing function $v \colon A \to \mathbb{R}$,

$$\sum_{a \in A} p(a)v(a) \geq \sum_{a \in A} q(a)v(a).$$

(ii) Show that first-order stochastic dominance implies second-order stochastic dominance. Give an example of two lotteries $p, q \in \mathcal{L}(A)$ so that p second-order stochastically dominates q but p does not first-oder stochastically dominate q.

(iii) Based on the definitions of FOSD and SOSD, guess a definition of third-order stochastic dominance.

5 Subjective Expected Utility

In Section 4, we considered lotteries where the probabilities for the alternatives are given. This postulates an objective notion of probability, that is, a canonical way of pinning down the probability of an event, or, even more ambitiously, the “true” probability. In some cases, one can sort of make sense of this, whereas in others, one cannot.

One possibility is the frequentist approach. If an experiment is repeated sufficiently often under very similar conditions, we may expect that the empirical distribution over its outcomes lets us assess the distribution over outcomes on the next repetition. In mathematically terms, if the experiments are i.i.d. random variables, the empirical frequencies almost surely converge to the “true” probabilities by the law of large numbers. This is what insurance companies use to estimate the probabilities of damage events, which in turn determine the customers’ premiums. But the frequentist approach fails if there are no or too few observations of sufficiently similar past experiments.

A second approach is arguing by symmetry. Suppose a six-sided die is symmetric with respect to shape, weight, and every other relevant physical quantity. If it is rolled with sufficient force, we may expect that each side has the same probability of showing up on top independently of the starting position. Formally, the idea behind symmetry is to partition the space of states determining the outcome of the experiment into equally likely events. In some cases, one can thus hope to predict the distribution of outcomes without any historical data. But many experiments lack obvious symmetries. Even just trimming one side of the die breaks the symmetry and invalidates the argument.

So what is the probability of an event in cases where the frequentist and symmetry approach fail? For example, what is the probability that your friend passes their next exam? You may know about past exams, but those were in different subjects by different examiners with a different amount of preparation by your friend. Also, there are no apparent symmetries that would allow you to come up with a partition into equally likely events. Nevertheless, you probably have at least some vague sense of what the probability of them passing should be. But what is this based on, and what does it even mean?

One answer to the second question is the subjectivist view of probability. It holds that probabilities reflect no more than the subjective beliefs of a decision-maker. In some cases, such as when rolling a fair die or assessing whether a customer will have a car accident based on historical data, the probabilities of most decision-makers agree, which gives the impression that there are objective probabilities. The subjectivist account can thus accommodate those two situations and give meaning to probability in the example of your friend’s exam.

The goal of this section is to examine when a decision-maker’s preferences over state-contingent acts are compatible with expected utility maximization according to some subjective belief. In contrast to Section 4, we thus cannot take probabilities as given but need to infer them from preferences. We model this by assuming that there is a set of states of the world and a set of outcomes (for example, monetary payoffs, lotteries over alternatives, or abstract alternatives). An act is a function from states to outcomes and the decision-maker can choose between acts. Under certain assumptions about her preferences over acts, she behaves as if maximizing her expected utility according to some probability distribution over states (and a utility function on alternatives). This subjective probability distribution is called her belief.

5.1 De Finetti's Model

Consider a decision-maker who can choose between acts whose prizes depend on the state of the world. The state is unknown to the decision-maker. To model this situation, let S be a finite set of states and $\mathbb{R}$ be the set of outcomes, interpreted as monetary prizes. An act is a function from states to outcomes, so that $\mathbb{R}^S$ is the set of acts. Hence, if the state were known, the outcome of every act would be known. For a belief $p \in \mathcal{L}(S)$, the expected payoff of an act $f \in \mathbb{R}^S$ is

$$\mathrm{E}_p(f) = \sum_{s \in S} p(s) f(s).$$

An example is a horse race, where a state might describe the physical condition of the horses and the abilities of the jockeys precisely enough to predict the winner, and acts are bets on the winner of the race.

Definition 5.1 (Expected payoff maximization)**.** A relation $\succsim$ on $\mathbb{R}^S$ maximizes expected payoff if there is a belief $p \in \mathcal{L}(S)$ so that E_p represents $\succsim$.

The problem of characterizing preferences represented by expected payoff maximization according to some subjective belief is dual to that of characterizing expected utility maximization over lotteries. In the latter case, the probabilities of the alternatives are given by lotteries and we had to identify the utilities for alternatives. Conversely, here the payoffs of acts are given and we have to determine the probabilities of the states. It turns out however that a similar set of axioms works.

Definition 5.2 (De Finetti's axioms)**.** A relation $\succsim$ over $\mathbb{R}^S$ satisfies de Finetti's axioms if

$$\succsim \text{ is complete and transitive,} \tag{F1}$$

$$\text{for all } f, g, h \in \mathbb{R}^S,\ f \succsim g \text{ implies } f + h \succsim g + h, \tag{F2}$$

$$\text{for all } f \in \mathbb{R}^S,\ U_{\succsim}(f) \text{ and } L_{\succsim}(f) \text{ are open,} \tag{F3}$$

$$\text{for all } f, g \in \mathbb{R}^S,\ f \geq g \text{ implies } f \succsim g, \text{ and} \tag{F4}$$

$$\text{there exist } f, g \in \mathbb{R}^S \text{ with } f \succ g. \tag{F5}$$

F1–F3 are analogs of vNM1–vNM3 used in the characterization of expected utility. F1 prescribes that $\succsim$ is a preference relation and is literally the same as vNM1. F2 is called translation-invariance. It states that the preference between two acts should not change if a third act is added to both of them. It thus has a similar flavor as the independence axiom vNM2, and, like the latter, implies that indifference curves are straight parallel lines. In the horse race example, it stipulates that whether or not you place an additional bet should not influence your preferences between two bets. The special case in which h is a constant act thus states that preferences a whealth-independent, which is a strong assumption. F3 is a topological notion of continuity, which requires that upper and lower contour sets are open. Heuristically, this means that a strict preference is robust to small perturbations of the payoffs. By contrast, vNM3 is an algebraic notion of continuity. F4 is monotonicity in the sense that f is preferred to g if f pays more than g in every state.[7] It is analogous to MON, which also demands a preference for higher sure prizes. F5 is non-triviality. The only preference relation violating this axiom is, of course, the trivial one (indifference between all acts). Note that non-triviality cannot be invalidated empirically.

The ensuing result shows that the expected payoff maximizing relations are precisely those satisfying the five axioms above.

Theorem 5.3 (de Finetti, 1937)**.** *A relation $\succsim$ on $\mathbb{R}^S$ satisfies F1–F5 if and only if it is non-trivial and maximizes expected payoff for some belief $p \in \mathcal{L}(S)$. Concretely, $\succsim$ satisfies F1–F5 if and only if it is non-trivial and there is $p \in \mathcal{L}(S)$ so that for all $f, g \in \mathbb{R}^S$,*

$$f \succsim g \text{ if and only if } E_p(f) \geq E_p(g).$$

In that case, p is unique.

For $S' \subset S$, let $\mathbf{1}_{S'}$ be the indicator function of S'. Moreover, let $\mathbf{1} = \mathbf{1}_S$ and $\mathbf{0} = 0 \cdot \mathbf{1}$ be the acts with payoff 1 and 0, respectively, in every state. The following lemmas will be used several times in the proof of Theorem 5.3.

Lemma 5.4. *Let $\succsim$ be a relation on $\mathbb{R}^S$ satisfying F1 and F2. Then, for all $f, f', g, g' \in \mathbb{R}^S$,*

$$f \succsim g \text{ and } f' \succsim g' \text{ implies } f + f' \succsim g + g'.$$

If one of the preferences in the antecedent is strict, so is the preference in the consequent. Moreover, if $\succsim$ additionally satisfies F3, then $f \sim g$ implies $\alpha \cdot f \sim \alpha \cdot g$ for all $\alpha \in \mathbb{R}$.

Proof. Suppose $f \succsim g$ and $f' \succsim g'$. By F2, we have $f + f' \succsim g + f'$ and $g + f' \succsim g + g'$. Since $\succsim$ is transitive by F1, it follows that $f + f' \succsim g + g'$, and the preference is strict if one of the two preferences above is strict.

To prove the second part, suppose $f \sim g$. By repeated application of the first part, $k \cdot f \sim k \cdot g$ for all $k \in \mathbb{N}$. Now let $\alpha \in \mathbb{R}_+$ and assume for contradiction that $\alpha \cdot f \succ \alpha \cdot g$. Since $\succsim$ is continuous by F3, there is $\epsilon > 0$ so that $\alpha' \cdot f \succ \alpha' \cdot g$ for all $\alpha' \in \mathbb{R}$ with $|\alpha - \alpha'| < \epsilon$. Let $\alpha' \in \mathbb{Q}_+$ so that $|\alpha - \alpha'| < \epsilon$, and write $\alpha' = \frac{k}{l}$ for $k, l \in \mathbb{N}$. Then, $\frac{k}{l} \cdot f \succ \frac{k}{l} \cdot g$. Repeated application of the first part gives $k \cdot f \succ k \cdot g$, which is a contradiction. Hence, $\alpha \cdot f \sim \alpha \cdot g$ for all $\alpha \in \mathbb{R}_+$. Lastly, if $\alpha \in \mathbb{R}_-$ and $\alpha \cdot f \succ \alpha \cdot g$, we get by the first part of the lemma that

$$\mathbf{0} = \alpha \cdot f + (-\alpha) \cdot f \succ \alpha \cdot g + (-\alpha) \cdot g = \mathbf{0}.$$

This is again a contradiction. ∎

[7] For $f, g \in \mathbb{R}^S$, $f \geq g$ if $f(s) \geq g(s)$ for all $s \in S$.

Lemma 5.5. *Let $\succsim$ be a relation on $\mathbb{R}^S$ satisfying F1–F5. Then, for all $\alpha, \beta \in \mathbb{R}$ with $\alpha > \beta$, $\alpha \cdot \mathbf{1} \succ \beta \cdot \mathbf{1}$.*

Proof. By F2 (applied with $h = \beta \cdot \mathbf{1}$), it suffices to show that $\alpha \cdot \mathbf{1} \succ \mathbf{0}$ for all $\alpha > 0$. To this end, let $\alpha > 0$. Monotonicity implies that $\alpha \cdot \mathbf{1} \succsim \mathbf{0}$. Assume for contradiction that $\alpha \cdot \mathbf{1} \sim \mathbf{0}$. Lemma 5.4 then implies that $\gamma \cdot \mathbf{1} \sim 0$ for all $\gamma \in \mathbb{R}$.

On the other hand, since $\succsim$ is non-trivial by F5, there are $f, g \in \mathbb{R}^S$ so that $f \succ g$. F2 (applied with $h = -g$) gives $f - g \succ \mathbf{0}$. Hence, by transitivity of $\succsim$, $f - g \succ \gamma \cdot \mathbf{1}$ for all $\gamma \in \mathbb{R}$. But $\gamma \cdot \mathbf{1} \geq f - g$ for γ large enough, in which case $\gamma \cdot \mathbf{1} \succsim f - g$ by monotonicity, which is a contradiction. $\square$

Proof of Theorem 5.3. Showing that every expected payoff maximizing relation satisfies the axioms is straightforward. We omit the proof.

Conversely, assume that $\succsim$ satisfies F1–F5. The first step is constructing a candidate belief $p \in \mathcal{L}(S)$. For every $S' \subset S$, monotonicity gives that $\mathbf{1} \succsim \mathbf{1}_{S'} \succsim \mathbf{0}$. Let $I = \{\alpha \cdot \mathbf{1} \colon \alpha \in [0,1]\}$. By F3, $U(\mathbf{1}_{S'})$ and $L(\mathbf{1}_{S'})$ are open, and so the union of $U(\mathbf{1}_{S'}) \cap I$ and $L(\mathbf{1}_{S'}) \cap I$ cannot be all of I. Thus, there is $\alpha \in [0,1]$ so that $\mathbf{1}_{S'} \sim \alpha \cdot \mathbf{1}$. Now for $s \in S$, let $\alpha_s \in [0,1]$ so that $\mathbf{1}_{\{s\}} \sim \alpha_s \cdot \mathbf{1}$ and define $p \in \mathcal{L}(S)$ by letting $p(s) = \alpha_s$.

To ensure that p is well-defined, we have to show that α_s is unique and $\sum_{s \in S} \alpha_s = 1$. Uniqueness of α_s follows directly from Lemma 5.5. Moreover, repeated application of the first part of Lemma 5.4 gives

$$\mathbf{1} = \sum_{s \in S} \mathbf{1}_{\{s\}} \sim \sum_{s \in S} \alpha_s \cdot \mathbf{1}.$$

Hence, $\sum_{s \in S} \alpha_s = 1$ by Lemma 5.5.

It remains to show that E_p represents $\succsim$. For all $f \in \mathbb{R}^S$, we have by Lemma 5.4 that

$$f = \sum_{s \in S} f(s) \cdot \mathbf{1}_{\{s\}} \sim \sum_{s \in S} f(s)\alpha_s \cdot \mathbf{1} = \sum_{s \in S} f(s)p(s) \cdot \mathbf{1} = \mathrm{E}_p(f) \cdot \mathbf{1}.$$

Hence, by Lemma 5.5, for all $f, g \in \mathbb{R}^S$, $f \succsim g$ if and only if $\mathrm{E}_p(f) \geq \mathrm{E}_p(g)$. The uniqueness of p follows from observing that any two distinct beliefs induce different preferences. $\square$

5.2 Anscombe and Aumann's Model

In the model of de Finetti, the preferences over acts are pinned down solely by a belief about the states. This is because the outcomes of acts are monetary prizes and expected payoff maximizing preferences are considered. The two models in this and the next section allow for a richer set of outcomes. In that case, a natural class of preferences are those that maximize expected utility for some belief about the states and some utility function on outcomes.

The approach of Anscombe and Aumann (1963) is to distinguish between subjective uncertainty and objective uncertainty, or, in their terminology, between horse lotteries and roulette lotteries. For a horse race, it is a priori not only unclear who will win but different individuals may even have different probabilistic beliefs about it (depending, for example, on their information about the race or how they process this information). By contrast, when spinning a well-made roulette wheel, while it is unclear which number will be selected, different individuals typically have the same belief about the distribution of numbers. Hence, uncertainty about horse races is subjective, while uncertainty about spins of roulette wheels may be treated as objective. Anscombe and Aumann model this as follows. There is a set of states of the world and a set of alternatives. The true state is unknown. An act assigns to every state a lottery over alternatives. In our example, the states are the possible winners of the horse race and the alternatives are the numbers on a roulette wheel.

More formally, let $S = \{1, \ldots, n\}$ be a set of states and A be a finite set of alternatives. An act is a function $f \colon S \to \mathcal{L}(A)$ from states to lotteries over alternatives. The set of acts is thus $\mathcal{L}(A)^S$. We identify a lottery $p \in \mathcal{L}(A)$ with the constant act f_p, where $f_p(s) = p$ for all $s \in S$. For two acts $f, g \in \mathcal{L}(A)^S$ and $\alpha \in [0, 1]$ the convex combination $\alpha f + (1 - \alpha)g$ is to be understood pointwise. That is, $\alpha f + (1-\alpha)g$ is the act that gives the lottery $\alpha f(s) + (1-\alpha)g(s)$ in state $s \in S$. As usual, $f\alpha g$ is short for $\alpha f + (1 - \alpha)g$. For a belief $\pi \in \mathcal{L}(S)$ and utility function $u \colon \mathcal{L}(A) \to \mathbb{R}$, the expected utility of an act $f \in \mathcal{L}(A)^S$ is

$$E_{\pi,u}(f) = \sum_{s \in S} \pi(s)u(f(s)).$$

Example 5.6 (Acts and expected utility)**.** Let $S = \{1, 2\}$ and $A = \{a, b, c\}$. Let f be the act with $f(1) = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$ and $f(2) = \frac{1}{3}\delta_a + \frac{1}{6}\delta_b + \frac{1}{2}\delta_c$. We can write f more compactly as a matrix with rows indexed by states and columns indexed by alternatives.

$$f = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}$$

For the belief $\pi = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$ and the utility function $u = (1, 0, -1)$, we have

$$E_{\pi,u}(f) = \pi(1) \cdot u(f(1)) + \pi(2) \cdot u(f(2)) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot (-\frac{1}{6}) = -\frac{1}{3}.$$

Definition 5.7 (Expected utility maximization)**.** A relation $\succsim$ on $\mathcal{L}(A)^S$ maximizes expected utility if there is a belief $\pi \in \mathcal{L}(S)$ and a linear utility function $u \colon \mathcal{L}(A) \to \mathbb{R}$ so that $E_{\pi,u}$ represents $\succsim$.

It is natural to ask when preferences can be represented as expected utility maximization according to some belief and some utility function. Anscombe and Aumann (1963) provide an answer to this question by combining the von Neumann-Morgenstern axioms with those of de Finetti. It is unsurprising that the resulting set of axioms does the job since the former allowed us to construct a linear utility function on lotteries, while the latter gave us a belief about the states.

Definition 5.8 (Anscombe Aumann axioms)**.** A relation $\succsim$ on $\mathcal{L}(A)^S$ satisfies the Anscombe-Aumann axioms if

$$\succsim \text{ is complete and transitive,} \tag{AA1}$$

$$\text{for all } f,g,h \in \mathcal{L}(A)^S \text{ and } \alpha \in (0,1),\ f \succsim g \text{ if and only if } f\alpha h \succsim g\alpha h, \tag{AA2}$$

$$\text{for all } f,g,h \in \mathcal{L}(A)^S \text{ with } f \succ g \succ h, \text{ there are } \alpha,\beta \in (0,1) \text{ with } f\alpha h \succ g \succ f\beta h, \tag{AA3}$$

$$\text{for all } f,g \in \mathcal{L}(A)^S \text{ with } f(s) \succsim g(s) \text{ for all } s \in S,\ f \succsim g, \text{ and} \tag{AA4}$$

$$\text{there exist } f,g \in \mathcal{L}(A)^S \text{ with } f \succ g. \tag{AA5}$$

All five axioms are already familiar from the preceding sections. AA1 requires $\succsim$ to be a preference relation and is the same as vNM1 and F1. AA2 is the independence axiom applied to acts rather than lotteries as in vNM2. AA3 is the same notion of continuity as vNM3. AA4 is worth a moment of contemplation. It states that if f is state-wise better than g, then f is preferred to g. The first part means that for every state s, the constant act $f(s)$ is preferred to the constant act $g(s)$. Hence, the axiom is the same monotonicity condition as F4 with the difference that outcomes of acts are now lotteries and not monetary payoffs. It turns out that AA4 is not as innocent as one might suspect (cf. Remark 5.12). Lastly, AA5 states that $\succsim$ is not complete indifference and is thus the same as F5. The main result of this section shows that these five axioms characterize expected utility maximization.

Theorem 5.9 (Anscombe and Aumann, 1963)**.** *A relation $\succsim$ on $\mathcal{L}(A)^S$ satisfies AA1–AA5 if and only if there exists a belief $\pi \in \mathcal{L}(S)$ and a non-constant linear utility function $u\colon \mathcal{L}(A) \to \mathbb{R}$ so that $E_{\pi,u}$ represents $\succsim$. Moreover, π is unique and u is unique up to positive affine transformations.*

The proof of Theorem 5.9 proceeds as follows. Observe that the set of acts $\mathcal{L}(A)^S$ is a convex subset of $\mathbb{R}^{S\times A}$. Hence, if $\succsim$ satisfies AA1–AA3, we can apply Theorem 4.6 and conclude that there is a linear function $v\colon \mathcal{L}(A)^S \to \mathbb{R}$ representing $\succsim$. A first lemma shows that v takes the form $v(f) = \sum_{s\in S} u_s(f(s))$, where each u_s is a linear function on $\mathcal{L}(A)$. In words, linear functions on acts are automatically additively separable with respect to the states. A second lemma concludes from monotonicity that the u_s are in fact all non-negative affine transformations of $u = \sum_{s\in S} u_s$. That is, $u_s = \alpha_s u + \beta_s$ for some $\alpha_s \in \mathbb{R}_+$ and $\beta_s \in \mathbb{R}$. Letting π be so that the probabilities are proportional to the α_s gives the desired representation of $\succsim$.

Lemma 5.10. *Let $v\colon \mathcal{L}(A)^S \to \mathbb{R}$ be a linear function. Then, for every $s \in S$, there is a linear function $u_s\colon \mathcal{L}(A) \to \mathbb{R}$ such that for all $f \in \mathcal{L}(A)^S$,*

$$v(f) = \sum_{s\in S} u_s(f(s)).$$

Proof. Assume that $v\colon \mathcal{L}(A)^S \to \mathbb{R}$ is linear. Without loss of generality, we may assume that there is $p_0 \in \mathcal{L}(A)$ so that $v(p_0) = 0$. For $p \in \mathcal{L}(A)$ and $s \in S$, let $\phi_{p,s}$ be the act with $\phi_{p,s}(s) = p$ and $\phi_{p,s'} = p_0$ for all $s' \neq s$. Define $u_s\colon \mathcal{L}(A) \to \mathbb{R}$ by letting $u_s(p) = v(\phi_{p,s})$ for all

$p \in \mathcal{L}(A)$. To see that u_s is linear, observe that for all $p, q \in \mathcal{L}(A)$ and $\alpha \in [0,1]$, we have

$$u_s(p\alpha q) = v(\phi_{p\alpha q,s}) = v(\phi_{p,s}\alpha\phi_{q,s}) = \alpha v(\phi_{p,s}) + (1-\alpha)v(\phi_{q,s}) = \alpha u_s(p) + (1-\alpha)u_s(q).$$

The third equality uses that v is linear.

Now let $f \in \mathcal{L}(A)^S$ and observe that

$$\frac{n-1}{n}p_0 + \frac{1}{n}f = \frac{1}{n}\sum_{s\in S}\phi_{f(s),s}.$$

Thus, since $v(p_0) = 0$, we get

$$\begin{aligned}\frac{1}{n}v(f) &= \frac{n-1}{n}v(p_0) + \frac{1}{n}v(f) = v(\frac{n-1}{n}p_0 + \frac{1}{n}f) = v(\frac{1}{n}\sum_{s\in S}\phi_{f(s),s}) \\ &= \frac{1}{n}\sum_{s\in S}v(\phi_{f(s),s}) = \frac{1}{n}\sum_{s\in S}u_s(f(s)).\end{aligned}$$

Hence, in summary, $v(f) = \sum_{s\in S} u_s(f(s))$ as required. $\square$

Lemma 5.11. *Let $v\colon \mathcal{L}(A)^S \to \mathbb{R}$ be a non-constant linear function, and for all $s \in S$, let u_s be as in Lemma 5.10. Assume that for all $f, g \in \mathcal{L}(A)^S$, if $v(f(s)) \geq v(g(s))$ for all $s \in S$, then $v(f) \geq v(g)$. Then, $u = \sum_{s\in S} u_s$ is non-constant and linear, and, for every $s \in S$, there are $\alpha_s \in \mathbb{R}_+$ and $\beta_s \in \mathbb{R}$ with $u_s = \alpha_s u + \beta_s$.*

Proof. First, assume for contradiction that u is constant. Then, for all $p, q \in \mathcal{L}(A)$, we have that

$$v(p) = \sum_{s\in S} u_s(p) = u(p) = u(q) = \sum_{s\in S} u_s(q) = v(q).$$

Hence, for any two acts $f, g \in \mathcal{L}(A)^S$, $v(f(s)) = v(g(s))$ for all $s \in S$. By assumption, this implies $v(f) = v(g)$, which contradicts that v is non-constant.

Now let $s \in S$ and assume for contradiction that there are no $\alpha_s \in \mathbb{R}_+$ and $\beta_s \in \mathbb{R}$ with $u_s = \alpha_s u + \beta_s$. So u and u_s do not represent the same preferences over $\mathcal{L}(A)$. Moreover, u_s is non-constant as otherwise we could choose $\alpha_s = 0$. Thus, there are $p, q \in \mathcal{L}(A)$ with $u(p) \geq u(q)$ and $u_s(p) < u_s(q)$. Let $f, g \in \mathcal{L}(A)^S$ with $f(s') = g(s')$ for all $s' \neq s$, $f(s) = p$, and $g(s) = q$. Then, for all $s' \neq s$,

$$v(f(s')) = u(f(s')) = u(g(s')) = v(g(s'))$$

and

$$v(f(s)) = v(p) = u(p) \geq u(q) = u(g(s)) = v(g(s)).$$

Thus, by assumption, $v(f) \geq v(g)$. On the other hand,

$$v(f) - v(g) = \sum_{s'\in S} u_{s'}(f(s')) - u_{s'}(g(s')) = u_s(p) - u_s(q) < 0.$$

This is a contradiction. Hence, u_s is a non-negative affine transformation of u as required. $\square$

Proof of Theorem 5.9. Checking the if part is straightforward. We prove the only if part. Assume that $\succsim$ satisfies AA1–AA5. AA1–AA3 are vNM1–vNM3 on the convex subset $\mathcal{L}(A)^S$ of $\mathbb{R}^{S\times A}$. Thus, by Theorem 4.6, there is a linear function $v\colon \mathcal{L}(A)^S \to \mathbb{R}$ representing $\succsim$. By Lemma 5.10, there are linear functions $u_s\colon \mathcal{L}(A) \to \mathbb{R}$ for $s \in S$ such that for all $f \in \mathcal{L}(A)^S$,

$$v(f) = \sum_{s\in S} u_s(f(s)).$$

By AA5, v is non-constant. Moreover, by AA4, for all $f, g \in \mathcal{L}(A)^S$, if $v(f(s)) \geq v(g(s))$ for all $s \in S$, then $v(f) \geq v(g)$. Hence, by Lemma 5.11, $u = \sum_{s\in S} u_s$ is non-constant and linear, and, for every $s \in S$, there are $\alpha_s \in \mathbb{R}_+$ and $\beta_s \in \mathbb{R}$ with $u_s = \alpha_s u + \beta_s$. Since u is non-constant, not all α_s are 0. Let $\alpha = \sum_{s\in S} \alpha_s$ and $\beta = \sum_{s\in S} \beta_s$, and define a belief $\pi \in \mathcal{L}(S)$ by letting for every $s \in S$,

$$\pi(s) = \frac{\alpha_s}{\alpha}.$$

Then, for all $f \in \mathcal{L}(A)^S$, we have

$$v(f) = \sum_{s\in S} u_s(f(s)) = \sum_{s\in S} \alpha_s u(f(s)) + \beta_s = \alpha \cdot \sum_{s\in S} \pi(s)u(f(s)) + \beta = \alpha \cdot \mathrm{E}_{\pi,u} + \beta.$$

It follows that $\mathrm{E}_{\pi,u}$ represents $\succsim$.

Lastly, assume that are beliefs π, π' and linear utility functions u, u' so that $E_{\pi,u}$ and $E_{\pi',u'}$ both represent $\succsim$. Since the set of constant acts can be identified with $\mathcal{L}(A)$ and the preferences over constant acts do not depend on the belief, it follows from Theorem 4.4 that u is unique up to positive affine transformations. Thus, $E_{\pi,u}$ and $E_{\pi',u}$ represent the same preferences. But this can only be if $\pi = \pi'$. $\square$

Remark 5.12 (State-dependent utility). In the proof of Theorem 5.9, we first constructed a utility function u_s for every state s, and then, using the monotonicity axiom AA4, showed that all u_s are equal (up to positive affine transformations). If one dispenses with monotonicity, utilities may be state-dependent. State-dependent utilities are often sensible in practice since the value of an alternative sometimes does depend on the state. We discuss this more in the next section.

5.3 Savage's Model

The distinction between subjective and objective uncertainty in the model of Anscombe and Aumann requires classifying all uncertainty into these two categories. But where to draw the boundary may not always be obvious and can seem arbitrary. For example, if one picks a random digit in the decimal expansion of π, is the chosen digit subjectively or objectively uncertain?[8]

[8] Of course, there is no such thing as a random digit in an infinite decimal expansion. What is meant is the following. Suppose one chooses a digit uniformly at random among the first $N \in \mathbb{N}$ digits of the decimal expansion. Then, the probability that it equals $k \in \{0, 1, \ldots, 9\}$ is well-defined. Assuming this probability converges as N goes to infinity, we can make sense of probabilities for random digits in infinite expansions.

Every $k \in \{0, 1, \dots, 9\}$ has an objectively correct probability of being the chosen digit. Number theorists suspect that the probability is $\frac{1}{10}$ for all k but no one has managed to prove this yet. So are random digits of π subjectively or objectively uncertain?

The model of Savage (1954) avoids this issue by taking a purely subjectivist approach to probability. That is, it does not postulate an objectively correct probability for any event. Hence, all uncertainty is subjective. This approach was a conceptual leap forward from the state of the art at the time. (Note that Savage's model precedes that of Anscombe and Aumann.)

Let S be a set of states and A a finite set of alternatives. An act is a function $f\colon S \to A$ from states to alternatives, so that the set of acts is A^S. Thus, a state can be thought of as a description of the world so precise that if it were known, all uncertainty would be resolved and the outcome of every act known. (Note the difference to Anscombe and Aumann's model where acts map to lotteries over alternatives.) For $a \in A$, we write f_a for the constant act that gives a in every state. Sometimes we will abuse notation and write a instead of f_a. A subset E of S is called an event. For two acts f, g and an event E, we write fEg for the act that equals f on E and g on $S \setminus E$. That is, $(fEg)(s) = f(s)$ if $s \in E$ and $(fEg)(s) = g(s)$ if $s \in S \setminus E$. This operation can be thought of as the analog of taking convex combinations of acts in the preceding sections.

A function $\pi\colon 2^S \to \mathbb{R}$ is called a set function. It is finitely additive if for all pairwise disjoint $E_1, E_2, \dots, E_n \subset S$,

$$\pi\left(E_1 \cup E_2 \cup \dots \cup E_n\right) = \pi(E_1) + \pi(E_2) + \dots + \pi(E_n). \qquad \text{(finite additivity)}$$

Note that if π satisfies the above equality for $n = 2$, it follows by induction that π is finitely additive. We say that π is countably additive if for any sequence of pairwise disjoint $E_1, E_2, \dots$,

$$\pi\left(\bigcup_{i \geq 1} E_i\right) = \lim_{n \to \infty} \sum_{i \leq n} \pi(E_i). \qquad \text{(countable additivity)}$$

Note that countable additivity implies finite additivity since all but finitely many E_i may be empty. A probability measure on S is a countably additive set function $\pi\colon 2^S \to \mathbb{R}_+$ with $\pi(S) = 1$. If π is only finitely additive, it is a finitely additive probability measure. Denote by $\mathcal{F}(S)$ the set of finitely additive probability measures on S, and by $\mathcal{P}(S) \subset \mathcal{F}(S)$ the set of probability measures. A finitely additive probability measure π is non-atomic if for every $E \subset S$ with $\pi(E) > 0$, there is $F \subset E$ so that $0 < \pi(F) < \pi(E)$. Moreover, π is convex if for all $E \subset S$ and $\alpha \in [0, 1]$, there is $F \subset E$ so that $\pi(F) = \alpha\pi(E)$. Note that convexity implies non-atomicity. For countably additive probability measures, both properties are equivalent. We call finitely additive probability measures on S beliefs.

Example 5.13 (Finitely but not countably additive probability measures)**.** We give an example of a probability measure that is finitely additive but not countably additive. Let $S = \mathbb{N}$ and $\mathcal{U}$ be an ultrafilter on 2^S and let $\pi(E) = 0$ if $E \notin \mathcal{U}$ and $\pi(E) = 1$ if $E \in \mathcal{U}$.[a] The properties of an ultrafilter ensure that π is a finitely additive probability measure.

[a] $\mathcal{U} \subset 2^S$ is an ultrafilter if (i) $\emptyset \notin \mathcal{U}$, (ii) if $E \subset F \subset S$ and $E \in \mathcal{U}$, then $F \in \mathcal{U}$, (iii) if $E, F \in \mathcal{U}$, then $E \cap F \in \mathcal{U}$, and (iv) for all $E \subset S$, either $E \in \mathcal{U}$ or $S \setminus E \in \mathcal{U}$. Using Zorn's lemma, one can show that ultrafilters exist.

A belief $\pi \in \mathcal{F}(S)$ and an act $f \in A^S$ induce a (finitely additive) lottery on alternatives, denoted by $\pi_* f$, as follows. For all $a \in A$,

$$(\pi_* f)(a) = \pi(f^{-1}(a)).$$

The lottery $\pi_* f$ is called the push-forward of π along f. For a belief $\pi \in \mathcal{F}(S)$ and a linear utility function $u \colon \mathcal{L}(A) \to \mathbb{R}$, the expected utility of an act $f \in A^S$ is

$$E_{\pi,u}(f) = u(\pi_* f) = \sum_{a \in A} \pi(f^{-1}(a)) u(\delta_a).$$

Remark 5.14 (σ-algebras and measurable functions). In general, a probability measure π on S is a set function from a σ-algebra $\mathcal{S} \subset 2^S$ on S to $\mathbb{R}$. In that case, for push-forwards to be well-defined, one needs to assume that acts are measurable functions. We avoid these technicalities by implicitly assuming that $\mathcal{S} = 2^S$.

Example 5.15 (Acts and expected utility)**.** Let $S = \mathbb{N}$ and $A = \{a, b\}$. Let f be the act with $f(s) = a$ if s is even and $f(s) = b$ if s is odd. Denoting by $E \subset \mathbb{N}$ the set of even numbers, we can write $f = f_a E f_b$. For the belief π with $\pi(s) = 2^{-s}$ for all $s \in S$, we have $(\pi_* f)(a) = 2^{-2} + 2^{-4} + \cdots = \frac{1}{3}$ and $(\pi_* f)(b) = 2^{-1} + 2^{-3} + \cdots = \frac{2}{3}$. Hence, for the linear utility function $u \colon \mathcal{L}(A) \to \mathbb{R}$ with $u(\delta_a) = 1$ and $u(\delta_b) = 0$, we have

$$E_{\pi,u}(f) = u\left(\frac{1}{3}\delta_a + \frac{2}{3}\delta_b\right) = \frac{1}{3}.$$

Definition 5.16 (Expected utility maximization)**.** A relation $\succsim$ on A^S maximizes expected utility if there is a belief $\pi \in \mathcal{F}(S)$ and a linear utility function $u \colon \mathcal{L}(A) \to \mathbb{R}$ so that $E_{\pi,u}$ represents $\succsim$.

Suppose $\succsim$ is a non-trivial relation on acts that admits an expected utility representation with belief π and utility function u. Observe that if π assigns probability 0 to some event E, then $\succsim$ does not depend on the values of acts on E. Events with the latter property can also be defined without an expected utility representation and called null events. Considering null events will be useful when constructing the belief for an expected utility representation assuming π satisfies certain axioms.

Definition 5.17 (Null events)**.** Let $\succsim$ be a relation on A^S. An event $E \subset S$ is null for $\succsim$ if for all $f, g, h \in A^S$, $fEh \sim gEh$.

One of Savage's contributions is to provide a set of normatively justifiable axioms for relations on acts that is equivalent to expected utility maximization. In contrast to the axioms of de Finetti and Anscombe and Aumann, those are fundamentally different from those of von Neumann and Morgenstern. In fact, there is no obvious way to define vNM2 and vNM3 in Savage's model since it does not involve lotteries as primitives. But as we shall see, it is still possible to draw some parallels between Anscombe and Aumann's axioms and those of Savage.

Definition 5.18 (Savage's axioms)**.** A relation $\succsim$ on A^S satisfies Savage's axioms if

$$\succsim \text{ is complete and transitive,} \tag{S1}$$

$$\text{for all } f, g, h, h' \in A^S \text{ and } E \subset S,\ fEh \succsim gEh \text{ if and only if } fEh' \succsim gEh', \tag{S2}$$

$$\text{for all } a, b \in A,\ f \in A^S, \text{ and } E \subset S \text{ non-null},\ a \succsim b \text{ if and only if } aEf \succsim bEf, \tag{S3}$$

$$\begin{gathered}\text{for all } a, b, a', b' \in A \text{ with } a \succ b \text{ and } a' \succ b' \text{ and } E, F \subset S,\ aEb \succsim aFb \\ \text{if and only if } a'Eb' \succsim a'Fb',\end{gathered} \tag{S4}$$

$$\text{there are } f, g \in A^S \text{ with } f \succ g, \textit{and} \tag{S5}$$

$$\begin{gathered}\text{for all } f, g, h \in A^S \text{ with } f \succ g, \text{ there exists a partition } \{E_1, \dots, E_n\} \text{ with} \\ hE_if \succ g \text{ and } f \succ hE_ig \text{ for all } i \in [n].\end{gathered} \tag{S6}$$

We discuss Savage's axioms in some length. This part stays close to the corresponding discussion of Gilboa (2009, Chapter 10.3).

S1 is familiar from the preceding sections and we will not dwell further on it.

S2 states that the preference between two acts should not depend on the states for which they are equal. Suppose f and g agree off of the event E.

$$f(s) = g(s) \text{ for all } s \in S \setminus E$$

Moreover, suppose f' and g' are so that f' agrees with f on E and g' agrees with g on E and f' and g' agree off of E.

$$f(s) = f'(s) \text{ and } g(s) = g'(s) \text{ for all } s \in E \text{ and } f'(s) = g'(s) \text{ for all } s \in S \setminus E$$

Then, S2 implies that $f \succsim g$ if and only if $f' \succsim g'$. In other words, it yields well-defined conditional preferences. To see a practical example, assume you consider going to a concert. You will go if you get tickets and stay home otherwise. Since it may rain, you consider taking an umbrella with you if you get tickets. If you stay home, the umbrella can stay in the cloakroom. Hence, the two acts you are comparing are

f = take an umbrella if you get tickets and leave it in the cloakroom otherwise, and

g = do not take an umbrella if you get tickets and leave it in the cloakroom otherwise.

S2 says that for comparing f and g, one can ignore what happens when you do not get tickets. Put differently, it allows making statements like "conditional on getting tickets, you prefer f to

g (or vice versa)". Note that the states in the example are the four combinations resulting from getting or not getting tickets and rain or no rain.

S2 is often called Savage's sure thing principle. To see why, let us modify our example. Suppose that if you do not get tickets, instead of staying at home, you go for a walk in the park. Let f be the act that you take an umbrella with you no matter what and let g be the act of leaving the umbrella in the cloakroom no matter what. Let us assume that you prefer f to g conditional on getting tickets and going to the concert and also conditional on not getting tickets and going for a walk. So you prefer f to g whether or not you get tickets. Then, S2 (in conjunction with S1) implies that you prefer f to g unconditionally.

S3 states that a constant act a is preferred to a constant act b if and only if the same preference persists when changing both acts to the same act off of some non-null event E. Put differently, if a is preferred to b, then taking any act and modifying it so that it gives a for states where it gave b results in a more preferred act. (Note that the "if part" is not appealing when E is a null event since then $aEf \succsim bEf$ should not imply $a \succsim b$.)

The latter formulation suggests an interpretation of S3 as a monotonicity axiom as in de Finetti's and Anscombe and Aumann's model. From the observation that the constant act a is preferred to the constant act b, we infer that a is a "better" alternative than b. So replacing the b's on E by a's in the act bEf should give a more preferred act. Note that this assumes that the desirability of an alternative is state-independent. We discuss this objection to S3 later.

Observe that S3 (in conjunction with S1) is equivalent to the following condition. For all $a, b \in A$, $f \in A^S$, and non-null $E, F \subset S$, $aEf \succsim bEf$ if and only if $aFf \succsim bFf$. (To see the equivalence, observe that letting $F = S$ gives S3, and two applications of S3 give the condition.) If this condition were to be violated, we would have $aEf \succ bEf$ and $bFf \succsim aFf$. From the former, we would infer that a is better than b, whereas the latter suggests that b is at least as good as a. So we would not be able to construct well-defined preferences over alternatives, let alone a utility function.

S4 is the analog of S3 for events rather than alternatives. Suppose I want to infer from your preferences over acts which of two events E and F you consider more likely. To this end, I offer you two bets. The first, called f, pays \$1 if E occurs and \$0 otherwise. The second, called g, is the same with F instead of E. If you prefer f to g, I would infer that you think that E is more likely than F. Now we change the stakes and let f' pay \$10 if E occurs and \$5 otherwise. Again, g' is defined similarly with F instead of E. If you prefer g' to f', I would infer that you consider F to be more likely than E. Taking both observations together, I would not be able to infer a consistent ranking of events in terms of your subjective probability. S4 requires that this situation cannot occur. So $f \succsim g$ if and only if $f' \succsim g'$.

S5 is the same non-triviality axiom that is part of the axioms of de Finetti and Anscombe and Aumann. As in the latter case, the trivial preference relation admits an expected utility representation. Indeed, any belief and any constant utility function work. This severe non-uniqueness of the belief is the sole reason for imposing S5.

S6 can be seen as a continuity axiom. Since there is no inherent topology on acts as in the case when alternatives are payoffs or lotteries, we need to capture the same idea in a round-about way. We want so say that if $f \succ g$, then for every act f' sufficiently close to f, we have $f' \succ g$. In the models of de Finetti and Anscombe and Aumann we were able to say that two acts are close to each other if their outcomes (payoffs, lotteries over alternatives) are close to each other for every state. But we cannot do the same here since we do not have a notion of closeness for alternatives. Alternatively, we could say that the two acts should only differ on a small set of states. If we had a probability measure on S, we could say they are close if they differ only on a low probability event. We do not have such a probability measure, but we can reason backwards. Suppose we had a finitely additive probability measure π on S, and let us assume it has no large chunks, that is, that it is convex. Then, for every $\epsilon > 0$, we could partition S into events $E_1, E_2, \dots, E_n$ so that $\pi(E_i) < \epsilon$ for all i. S6 then requires that $f' \succ g$ if f' differs from f only on one of the E_i. It thus states that the preferences satisfy the notion of continuity inherent to expected utility maximizing preferences based on convex beliefs.

How demanding is S6? For a start, it requires that every state is null. To see this, let $s \in S$ and assume for contradiction that there are acts f, g so that $f \succ g$ and $f(s') = g(s')$ for all $s' \neq s$. Let $h = g$. By S6, there is a partition $\{E_1, \dots, E_n\}$ of S so that $hE_if \succ g$ for all i. In particular, if $s \in E_j$, $g = hE_jf \succ g$, which is a contradiction. Hence, $\{s\}$ is a null event. If S1 holds, the union of any two, and, thus, any finite number of null events is again null. So if S is finite, S itself is a null event, which is to say that $\succsim$ is trivial. In summary, we conclude that S1, S5, and S6 together require S to be infinite. One argument defending the assumption that S is infinite goes as follows. Let S' be some infinite state space, say, $S' = \{0,1\}^{\mathbb{N}}$ describes an infinite sequence of coin tosses. We can then augment S by splitting each state into infinitely many "substates" obtained from the sequence of coin tosses. That is, we replace S by $S \times S'$. This solves the problem of a finite state space and allows non-trivial preference relations to satisfy S6. But now we are faced with the task of extending a preference relation on A^S to one on $A^{S \times S'}$ so that S1–S5 remain satisfied, which may not be trivial.

Savage has shown that the six axioms are necessary and sufficient for a relation to admit an expected utility representation with a convex belief. As usual, proving necessity is straightforward. Sufficiency is much harder and a genuine mathematical achievement.

Theorem 5.19 (Savage, 1954)**.** *A relation $\succsim$ on A^S satisfies S1–S6 if and only if it admits an expected utility representation $\mathrm{E}_{\pi,u}$ with convex π. Moreover, π is unique and u is unique up to positive affine transformations.*

Savage's proof is rather long. More concise and accessible proofs have been given by Fishburn (1970) and Kreps (1988). We discuss the main steps of the proof, postponing details for the moment.

Step 1 (Constructing a qualitative probability). Define a relation $\succsim^*$ on 2^S as follows. Let $a, b \in A$ with $a \succ b$. For all $E, F \subset S$, let

$$E \succsim^* F \text{ if and only if } aEb \succsim aFb.$$

S4 ensures that $\succsim^*$ does not depend on the choice of a and b. We claim that $\succsim^*$ has the following properties.

$$\succsim^* \text{ is complete and transitive,} \tag{S1*}$$

$$\text{for all } E, F, G \subset S \text{ with } (E \cup F) \cap G = \emptyset,\ E \succsim^* F \text{ if and only if } E \cup G \succsim^* F \cup G, \tag{S2*}$$

$$\text{for all } E \subset S,\ E \succsim^* \emptyset, \text{ and} \tag{S3*}$$

$$S \succ^* \emptyset. \tag{S5*}$$

S1* follows from S1. S2 applied with $f = aEb$, $g = aFb$, $h = b$, and $h' = aGb$ gives S2*. S3 applied with $f = b$ gives S3*. (If E is null, S3* holds with indifference by the definition of null events.) Lastly, S5* follows from S5 and repeated application of S2.

Definition 5.20 (Qualitative probability)**.** A relation $\succsim^*$ on 2^S that satisfies S1*–S5* is a qualitative probability.

Note that if π is a probability measure, the relation $\succsim^*$ on 2^S given by $E \succsim^* F$ if and only if $\pi(E) \geq \pi(F)$ is a qualitative probability. But not every qualitative probability is represented by some probability measure. Kraft et al. (1959) provide a counter example.

Step 2 (S6 implies that $\succsim^*$ is represented by a probability measure). If $\succsim$ satisfies S6, then $\succsim^*$ satisfies an analogous continuity property.

$$\text{For all } E, F \subset S \text{ with } E \succ^* F, \text{ there exists a partition } \{E_1, \dots, E_n\} \text{ of } S \text{ with } E \succ^* (F \cup E_i) \text{ for all } i \in [n]. \tag{S6*}$$

If a qualitative probability satisfies S6*, it is represented by a unique convex finitely additive probability measure. The proof proceeds along the following steps.

(i) For every n, there exists an equipartition $\{E_1, \dots, E_{2^n}\}$. That is, $E_i \sim^* E_j$ for all $i, j \in [2^n]$.

(ii) Define
$$\pi(E) = \sup_n \max_{\{E_1,\dots,E_{2^n}\}} \frac{|\{i \colon E_i \subset E\}|}{2^n}$$
where the maximum is taken over all equipartitions of size 2^n. One can check that π is well-defined and a finitely additive probability measure.

(iii) The construction shows that π is unique. From the existence of equipartitions, one can deduce convexity.

Step 3 (Induced lotteries over outcomes). For an act f, let $p_f = \pi_* f \in \mathcal{L}(A)$. Convexity of π implies that the mapping $f \mapsto p_f$ from A^S to $\mathcal{L}(A)$ is surjective. Moreover, using S2*, one can show that $p_f = p_g$ implies $f \sim g$. This allows defining a relation $\succsim'$ on $\mathcal{L}(A)$ by letting $p_f \succsim' p_g$ if and only if $f \succsim g$. Then one shows that $\succsim'$ satisfies vNM1–vNM3. (S1 implies vNM1, S2 and convexity of π are used to verify vNM2, and S6 gives vNM3.) Hence, by Theorem 4.6, there is a linear function $u \colon \mathcal{L}(A) \to \mathbb{R}$, unique up to positive affine transformations, representing $\succsim'$.

Step 4. Together, the preceding steps imply that π and u represent $\succsim$ as in Theorem 5.19.

The following sequence of lemmas gives details on the steps above. We assume throughout that $\succsim^*$ satisfies S1*–S6*.

Lemma 5.21 (Compatibility with disjoint unions). *Let $E, F, E', F' \subset S$ with $E \cap F = \emptyset = E' \cap F'$. If $E \succ^* E'$ and $F \succ^* F'$, then $E \cup F \succ^* E' \cup F'$.*

Proof. We distinguish two cases. The idea in each case is to swap part of E' with part of F' to reduce to a simpler case.

Case 1 ($E' \setminus E \succsim^* E \cap F'$). By S5*, we can find a partition $\{E'_1, \dots, E'_n\}$ of $E' \setminus E$ so that $F \succ^* F' \cup E'_i$ for all $i \in [n]$. Let k be minimal so that $G = \bigcup_{i=1}^k E'_i \succsim^* E \cap F'$. Such k exists by assumption of this case. Let $\tilde{E} = (E' \setminus G) \cup (E \cap F')$ and $\tilde{F} = (F' \setminus E) \cup G$. We have $E \succ^* E' \succsim^* \tilde{E}$ by definition of G and $F \succ^* \tilde{F}$ by minimality of k. Moreover, $E' \cup F' = \tilde{E} \cup \tilde{F}$ and $E \cap \tilde{F} = \emptyset$. Hence, by S2*,

$$E \cup F \succ^* E \cup \tilde{F} \succ^* \tilde{E} \cup \tilde{F} = E' \cup F'.$$

Case 2 ($E \cap F' \succ^* E' \setminus E$). By S5*, we can find a partition $\{F'_1, \dots, F'_n\}$ of $E \cap F'$ so that $E \succ^* E' \cup F'_i$ for all $i \in [n]$. Let k be minimal so that $G = \bigcup_{i=1}^k F'_i \succsim^* E' \setminus E$. Such k exists by assumption of this case. Let $\tilde{E} = (E' \cap E) \cup G$ and $\tilde{F} = (F' \setminus G) \cup (E' \setminus E)$. We have $\tilde{E} \subset E$ and $F \succ^* F' \succsim^* \tilde{F}$ by definition of G. Moreover, $E' \cup F' = \tilde{E} \cup \tilde{F}$ and $E \cap \tilde{F} = \emptyset$. Hence, by S2*,

$$E \cup F \succsim^* \tilde{E} \cup F \succ^* \tilde{E} \cup \tilde{F} = E' \cup F'.$$

□

Lemma 5.21 shows that $\succsim^*$ is compatible with "addition" of disjoint sets. The next lemma shows that it is also compatible with "multiplication" of sets. Together, these lemmas give us some arithmetic on 2^S, which makes our lives much easier (but still not easy) for the rest of the proof. For $E, F \subset S$ and $n \in \mathbb{N}$, we write $E > nF$ if there is a partition $\{E_1, \dots, E_n\}$ of E so that $E_i \succ^* F$ for all $i \in [n]$; likewise $nF > E$ if there is a partition $\{E_1, \dots, E_n\}$ of E so that $F \succ^* E_i$ for all $i \in [n]$. Note that $E > F$ if and only if $E \succ^* F$. Moreover, for all E, F, there is n so that $nF > E$ by S6*. We establish two notions of transitivity of the relation $>$.

Lemma 5.22 (Compatibility with multiplication). *Let $E, F, G \subset S$. Then, if $nF > E$ and $E > nG$, then $F \succ^* G$. Moreover, if $E > mF$ and $F > nG$, then $E > mnG$.*

Proof. To prove the first part, assume for contradiction that $G \succsim^* F$. Let $\{F_1, \dots, F_n\}$ and $\{G_1, \dots, G_n\}$ be partitions of E so that $F \succ^* F_i$ and $G_j \succ^* G$ for all $i, j \in [n]$. $G \succsim^* F$ and S1* imply $G_j \succ^* F_i$ for all $i, j \in [n]$. This contradicts Lemma 5.21.

Let $\{E_1, \dots, E_m\}$ and $\{F_1, \dots, F_n\}$ be partitions of E and F, respectively, so that $E_i \succ^* F$ and $F_j \succ^* G$ for all $i \in [m]$ and $j \in [n]$. Assume that $F_j \succsim^* F_1$ for all $j \in [n]$. By S6*, there is $H \subset S$ with $F_1 \succ^* G \cup H$. Again by S6*, there is a partition $\{E^i_1, \dots, E^i_{m_i}\}$ of E_i so that $H \succ^* E^i_k$ for all $i \in [m]$ and $k \in [m_i]$. Using Lemma 5.21, one shows that for all $i \in [m]$, there is a partition $\{I_1^i, \ldots, I_n^i\}$ of the index set $\{1, \ldots, m_i\}$ so that for $G_j^i = \bigcup_{k \in I_j} E_k^i$, $F_j \succ^* G_j^i \succ^* G$ for $j < n$ and $G_n^i \succ^* G$. Hence, $\{G_j^i\}_{i \in [m], j \in [m]}$ is a partition of E into mn sets so that $G_j^i \succ^* C$ for all $i \in [m]$ and $j \in [n]$. □

Lemma 5.23 (Fine partitioning). *For all $E \subset S$, there is a partition $\{F_1, \ldots, F_n\}$ of E so that $E > 2F_j$ for all $j \in [n]$.*

Proof. By S6*, there is a partition $\{E_1, \ldots, E_m\}$ of E so that $E \succ^* E_i$ for all $i \in [m]$. Note that necessarily $m \geq 2$. Assume without loss of generality that $E_i \succsim^* E_1$ for all $i \in [m]$. Again by S6*, there is a partition $\{F_1, \ldots, F_n\}$ of E so that $E_1 \succ^* F_j$ for all $j \in [n]$. Hence, $E > 2F_j$ for all $j \in [n]$. □

Lemma 5.24 (Equipartitioning). *For any $E \subset S$, there is $F \subset E$ with $F \sim^* E \setminus F$.*

Proof. We construct a sequence of partitions $\mathcal{P}^0, \mathcal{P}^1, \ldots$ of E, $\mathcal{P}^k = \{E_1^k, \ldots, E_{n_k}^k\}$, so that $\mathcal{P}^k$ refines $\mathcal{P}^{k-1}$ and $E > 2^k E_i^k > \emptyset$ for all $i \in [n_k]$ and $k \geq 1$.[9] Let $\mathcal{P}^0 = \{E\}$. Having defined $\mathcal{P}^{k-1}$, we can, by Lemma 5.23, partition E_i^{k-1} into $\{F_1, \ldots, F_m\}$ so that $E_{i'}^{k-1} > 2F_j > \emptyset$ for all $i' \in [n_{k-1}]$ and $j \in [m]$. Letting $\mathcal{P}^k$ be the union of all these partitions of the E_i^{k-1} gives, by Lemma 5.22, a partition with the desired property.

For all $k \geq 0$ and $i \in [n_k]$, let $F_i^k = E_1^k \cup \cdots \cup E_i^k$. Let

$$F^k = \bigcup \left\{ F_i^k \colon E \setminus F_i^k \succ^* F_i^k \right\}.$$

That is, F^k is the union of the longest initial sequence of $\{E_1^k, \ldots, E_{n_k}^k\}$ that is less preferred than its complement in E. By construction, $F^{k-1} \subset F^k \subset F^{k-1} \cup E_i^{k-1}$ for some $i \in [n_{k-1}]$. Let $F = \bigcup_{k \geq 0} F^k$. Since $F \setminus F^{k-1} \subset E_i^{k-1}$ for some $i \in [n_{k-1}]$, $E > 2^{k-1}(F \setminus F^{k-1})$.

Assume for contradiction that $E \setminus F \succ^* F$. By S6* and Lemma 5.23, there is $G \subset E \setminus F$, $G > \emptyset$, so that $(E \setminus F) \setminus G \succ^* F \cup G$. Let $k \geq 0$ so that $G \succ^* E_i^k$ for all $i \in [n_k]$. Thus, since $F^k \subset F$, $(E \setminus F^k) \setminus E_i^k \succ^* F^k \cup E_i^k$ by S2*. This contradicts the definition of F^k.

Conversely, assume that $F \succ^* E \setminus F$. Similar to before, there is $G \subset F$, $G > \emptyset$, so that $F \setminus G \succ^* (E \setminus F) \cup G$. Let $k \in \mathbb{N}$ so that $G \succ^* F \setminus F^k$. Using S2* for the first and third relation, we get

$$F^k \succsim^* F \setminus G \succ^* (E \setminus F) \cup G \succsim^* E \setminus F^k.$$

By S1*, this contradicts the definition of F^k.

We conclude that $F \sim^* E \setminus F$. □

Lemma 5.24 allows us to define a finitely additive probability π on S. It follows from Lemma 5.24 that for each n, there is a partition $\{E_1, \ldots, E_{2^n}\}$ of S so that $E_i \sim^* E_j$ for all $i, j \in [2^n]$. Such a partition is called a 2^n-equipartition. For $E \subset S$, let

$$\pi_-(E) = \sup_n \max \left\{ \frac{k}{2^n} \colon \{E_1, \ldots, E_{2^n}\} \text{ is a } 2^n\text{-equipartition and } k \in [2^n] \text{ with } E \succsim^* \bigcup_{i=1}^k E_i \right\},$$

$$\pi_+(E) = \inf_n \min \left\{ \frac{k}{2^n} \colon \{E_1, \ldots, E_{2^n}\} \text{ is a } 2^n\text{-equipartition and } k \in [2^n] \text{ with } \bigcup_{i=1}^k E_i \succsim^* E \right\}.$$

[9] A partition $\{E_1, \ldots, E_m\}$ of E refines another partition $\{F_1, \ldots, F_n\}$ of E if for all E_i there is F_j with $E_i \subset F_j$.

Intuitively, we approximate E by unions of subsets of equipartitions.

Lemma 5.25 (Equality of upper and lower measure)**.** *For all $E \subset S$, $\pi_-(E) = \pi_+(E)$.*

Proof. First observe that $\pi_+(E) \geq \pi_-(E)$. Hence, it suffices to show the converse inequality. If $E \sim^* S$, then $\pi_-(E) = \pi_+(E) = 1$. So assume that $S \succ^* E$.

Assume for contradiction that $\pi_+(E) - \pi_-(E) > \epsilon > 0$. Let $n_- \geq 0$ so that there is a 2^{n_-} equipartition $\{E_1^-, \dots, E_{2^{n_-}}^-\}$ of S and $k_- \in [2^{n_-}]$ maximal with $E \succsim^* \bigcup_{i=1}^{k_-} E_i^-$ and $\pi_-(E) - \frac{k_-}{2^{n_-}} \leq \frac{\epsilon}{2}$. Similarly, let $n_+ \geq 0$ so that there is a 2^{n_+} equipartition $\{E_1^-, \dots, E_{2^{n_+}}^-\}$ of S and $k_+ \in [2^{n_+}]$ minimal with $\bigcup_{i=1}^{k_+} E_i^- \succsim^* E$ and $\frac{k_+}{2^{n_+}} - \pi_+(E) \leq \frac{\epsilon}{2}$. Passing to finer equipartitions, we may assume that $n_- = n_+ = n$, $\epsilon > 2^{-n}$, and $E_i^- = E_i^+ = E_i$ for all $i \in [2^n]$. Then, $k_+ - k_- \geq 2$. But $E \succsim^* \bigcup_{i=1}^{k_-+1} E_i$ contradicts the choice of k_-, and $\bigcup_{i=1}^{k_-+1} E_i \succsim^* E$ the choice of k_+. We conclude that $\pi_-(E) = \pi_+(E)$. □

Let $\pi(E) = \pi_-(E) = \pi_+(E)$. By Lemma 5.25, π is well-defined. We prove that π is a finitely additive probability measure.

Lemma 5.26 (Finitely additivity and convexity)**.** *π is a convex and finitely additive probability measure.*

Proof. Clearly, for all $E \subset S$, $0 \leq \pi(E) \leq 1$, and $\pi(S) = 1$. To prove finite additivity, let $E, F \subset S$ with $E \cap F = \emptyset$. It is straightforward to check from the definitions that $\pi_-(E \cup F) \geq \pi_-(E) + \pi_-(F)$ and $\pi_+(E \cup F) \leq \pi_+(E) + \pi_+(F)$. Then,

$$\pi(E \cup F) = \pi_-(E \cup F) \geq \pi_-(E) + \pi_-(F) = \pi_+(E) + \pi_+(F) \geq \pi_+(E \cup F) = \pi(E \cup F).$$

Thus, $\pi(E \cup F) = \pi(E) + \pi(F)$.

To prove that π is convex, let $E \subset S$. By considering equipartitions of E, we observe that for all $n \geq 0$ and $k \in [2^n]$, there is $F \subset E$ so that $\pi(F) = \frac{k}{2^n}\pi(E)$. A limit argument similar to that in the proof of Lemma 5.24 shows that for all $\alpha \in [0, 1]$, there is $F \subset E$ with $\pi(F) = \alpha\pi(E)$. □

Having defined a finitely additive probability measure π on S, we show that the preferences over acts only depend on the lotteries they induce on A by π.

Lemma 5.27 (Reduction to induced lotteries)**.** *For all $f, g \in A^S$, if $\pi_* f = \pi_* g$, then $f \sim g$.*

Proof. First assume that f and g only differ in their preimages of two alternatives. That is, assume that there are $a, b \in A$, $E, F, G \subset S$, and $h \in A^S$ so that $f = (aEb)Gh$ and $g = (aFb)Gh$. Since $\pi_* f = \pi_* g$, it follows that $\pi(E) = \pi(F)$. Thus, by construction of π, $E \sim^* F$, and so $aEb \sim aFb$. Observing that $aEb = (aEb)Gb$ and $aFb = (aFb)Gb$ and applying S2* (with $h' = b$) gives $f \sim g$.

Repeated application of this case and S1 give $f \sim g$ whenever $\pi_* f = \pi_* g$. □

Lemma 5.28 (Surjectivity of the push-forward)**.** *The mapping from acts to lotteries over alternatives given by $f \mapsto \pi_* f$ is onto.*

Proof. We need to show that for each $p \in \mathcal{L}(A)$, there is $f \in A^S$ with $\pi_* f = p$. This follows readily from the convexity of π. □

Define a relation $\succsim'$ on lotteries $\mathcal{L}(A)$ by letting

$$p \succsim' q \text{ if and only if there are } f, g \in A^S \text{ with } f \succsim g,\ \pi_* f = p, \text{ and } \pi_* g = q.$$

Lemma 5.28 shows that for any two lotteries p and q, we can find f and g as above. Lemma 5.27 ensures that the relation between p and q does not depend on which f and g we choose. Hence, $\succsim'$ is well-defined. We show that $\succsim'$ satisfies the von Neumann-Morgenstern axioms, which in turn implies that it admits a utility representation.

Lemma 5.29. *$\succsim'$ satisfies vNM1–vNM3.*

Proof. S1 clearly implies vNM1.

We prove that S2 implies the weaker version of vNM2 stated in Remark 4.11. Let $p, q, r \in \mathcal{L}(A)$ with $p \succ' q$ and $\alpha \in (0,1)$. Let $E \subset S$ be an event with $\pi(E) = \alpha$, which exists since π is convex. Again using convexity of π, we can find an act f so that $\pi(f^{-1}(a) \cap E) = \alpha p(a)$ and $\pi(f^{-1}(a) \cap (S \setminus E)) = (1-\alpha)p(a)$ for all $a \in A$. Similarly, let g so that $\pi(g^{-1}(a) \cap E) = \alpha q(a)$ and $\pi(g^{-1}(a) \cap (S \setminus E)) = (1-\alpha)q(a)$ for all $a \in A$. Lastly, let h so that $\pi(h^{-1}(a) \cap E) = \alpha r(a)$ and $\pi(h^{-1}(a) \cap (S \setminus E)) = (1-\alpha)r(a)$ for all $a \in A$. Then, $\pi_* f = p$, $\pi_* g = q$, and $\pi_* h = r$. Moreover, $\pi_*(fEh) = p\alpha r$ and $\pi_*(gEh) = q\alpha r$.

By definition of $\succsim'$, we have $f \succ g$. Hence, by S2, we have

$$f = fEf \succ gEf \text{ if and only if } fEg \succ gEg = g, \text{ and}$$
$$f = fEf \succ fEg \text{ if and only if } gEf \succ gEg = g.$$

Since $f \succ g$, at least one of the above is true. Assume it is the former, so that $fEg \succ gEg$. Then, by S2, $fEh \succ gEh$. By definition of $\succsim'$, we have $p\alpha r \succ q\alpha r$.

A similar construction as in the preceding paragraph shows that S6 implies S3. □

Proof of Theorem 5.19. Since $\succsim'$ satisfies the prerequisites of Theorem 4.6, that theorem implies that there is a linear utility function $u\colon \mathcal{L}(A) \to \mathbb{R}$ representing $\succsim'$. That is,

$$p \succsim' q \quad \text{if and only if} \quad u(p) \geq u(q).$$

For any two acts $f, g \in A^S$, we thus have

$$\begin{aligned} f \succsim g \quad & \text{if and only if} \quad \pi_* f \succsim' \pi_* g \\ & \text{if and only if} \quad u(\pi_* f) \geq u(\pi_* g) \\ & \text{if and only if} \quad \sum_{a \in A} \pi(f^{-1}(a)) u(\delta_a) \geq \sum_{a \in A} \pi(g^{-1}(a)) u(\delta_b) \\ & \text{if and only if} \quad \mathrm{E}_{\pi,u}(f) \geq \mathrm{E}_{\pi,u}(g), \end{aligned}$$

where the first equivalence follows from the definition of $\succsim'$, the second from the fact that u represents $\succsim'$, and the third from linearity of u. This proves Theorem 5.19. □

Savage's result can be extended to infinite sets of alternatives. We discuss two possibilities. For this part, we drop the assumption that A is finite.

The first approach requires minimal technical adaptations. We say that an act f is simple if its image is finite. That is, $\{a \in A\colon$ there is $s \in S$ with $f(s) = a\}$ is finite. For relations over simple acts, Theorem 5.19 remains valid. Hence, a relation $\succsim$ on simple acts satisfies S1–S6 if and only if it can be represented by $E_{\pi,u}$ for some belief π and linear utility function u. The proof remains true up to minimal changes.

The second approach is more delicate. First, it requires an additional axiom.

$$\text{For all } f,g,h \in A^S \text{ and } E \subset S, \text{ if } fEh \succsim g(s)Eh \text{ for all } s \in E, \text{ then } fEh \succsim gEh, \text{ and} \\ \text{if } g(s)Eh \succsim fEh \text{ for all } s \in E, \text{ then } gEh \succsim fEh. \tag{S7}$$

For the case $E = S$, S7 states that if f is preferred to every alternative that g can give, then f should be preferred to g. The same logic extends to arbitrary E. It is far from obvious that S7 is not implied by S1–S6, but Savage (1954) gives an example proving that is it not. Second, the conclusion also changes. The representation now uses a (countably additive) probability measure instead of a finitely additive one.

Theorem 5.30 (Savage, 1954)**.** *A relation $\succsim$ on A^S satisfies S1–S7 if and only if it admits an expected utility representation $\mathrm{E}_{\pi,u}$ with non-atomic π and bounded u. Moreover, π is unique and u is unique up to positive affine transformations.*

Note that u needs to be bounded. For finite A, there was no need to state this since any function on a finite set is bounded. But what would go wrong if we allows u to be unbounded? In that case, we could construct an act, say, f, with infinite expected utility. The St. Petersburg paradox gives an example. Then, for all $a, b \in A$ and a suitable non-null event E, the expected utility of either of the two acts aEf and bEf would still be infinite, and so $aEf \sim bEf$. So by S3, $a \sim b$. But this need not be the case. An an extension of Savage's theorem to unbounded utility functions is provided by Wakker (1993).

Remark 5.31 (S3 is redundant for Theorem 5.30). Hartmann (2020) showed that S3 is redundant for Theorem 5.30. On the other hand, Savage (1954) showed that S1–S6 are all necessary for Theorem 5.19 and that all axioms except S4 are necessary for Theorem 5.30.

A conceptual difficulty of Savage's framework is that it is not always obvious how to choose the set of states. Indeed, a misspecified state space can lead to puzzling situations. As a case in point, let us consider Newcomb's paradox. There are two boxes on a table, one transparent and one opaque. The transparent box contains \$1,000. The opaque box may of may not contain \$1 million. You can take either both boxes (greedy) or only the opaque box (modest). Here is a first attempt of modeling this situation: there are two states, say, \$0 and \$1 million, depending on the content of the opaque box, and four outcomes, say, \$0, \$1,000, \$1,000,000, and \$1,001,000, depending on the amount of money you walk away with.

	\$0	\$1 million
greedy	\$1,000	\$1,001,000
modest	\$0	\$1,000,000

Being greedy is a dominant strategy and you should take both boxes. But wait, you have observed 1,000 people before you facing the same situation, half of which were greedy and half of which were modest. All the greedy ones walked away with \$1,000, whereas modesty was rewarded with \$1 million every single time.[10] Now you are torn. On the one hand, causal reasoning tells you that you should take both boxes because at the time of your decision, the content of the opaque box has already been determined. On the other hand, you have your Bayesian statistics down pat and find it highly unlikely that the probability for the opaque box to contain \$1 million does not depend on whether the decision-maker is greedy or not. More likely, the person who sets up the experiment is a conjurer or a mind-reader. So it seems Savage's model is not equipped to handle this situation since it suggests that any expected utility-maximizing decision-maker (who prefers more money to less) should be greedy, while this does not seem like the smart thing to do.

The problem comes from a misspecified state space. Our current model simply does not allow for the fact that the choice of the decision-maker influences the content of the opaque box. Instead, we should choose the state space to consist of all mappings from available options to outcomes. This allows the chosen option to influence the outcome in any arbitrary way. So in Newcomb's paradox, S should consist of all functions from $\{\text{greedy}, \text{modest}\}$ to $\{\$0, \$1{,}000, \$1{,}000{,}000, \$1{,}001{,}000\}$, which gives 16 states. Since each option can only result in two outcomes, some of these states can be eliminated (or rather, they are null states by the description of Newcomb's paradox). We are then left with four states, which can be expressed as follows.

$\emptyset$:	the opaque box never contains \$1 million
m:	the opaque box contains \$1 million when the decision-maker is modest
g:	the opaque box contains \$1 million when the decision-maker is greedy
$g \wedge m$:	the opaque box always contains \$1 million

Now the decision-problem becomes the following.

	$\emptyset$	m	g	$g \wedge m$
greedy	\$1,000	\$1,000	\$1,001,000	\$1,001,000
modest	\$0	\$1,000,000	\$0	\$1,000,000

[10]The standard formulation of Newcomb's paradox replaces the past observations by an omniscient being that knows in advance what you will choose and fills the opaque box accordingly. Our formulation follows Gilboa (2009) to avoid arguments about the existence and meaning of omniscience.

Observe that being greedy is no longer a dominant act since it is worse than being modest in state m. Based on the observations of preceding choices and the resulting outcomes, it seems reasonable to assign a very high probability to m in which case we would be well-advised to be content with the opaque box.[11]

The final point we discuss is that of state-dependent utilities. Savage's axioms give a representation of preferences by a subjective belief and a utility function. This representation assumes that the decision-maker derives the same utility from a given alternative in all states. However, one can come up with examples where utility arguably does depend on the state. Such cases ask for a more general model.

A state-dependent utility function $u\colon S \times A \to \mathbb{R}$ assigns a utility value to every alternative in every state. The expected utility of an act f is then

$$E_{\pi,u} = \sum_{s\in S} \pi(s)u(s, f(s)).$$

One problem with state-dependent utilities is that the representation is no longer unique in any useful way. In particular, any perturbation of the belief that does not change its support can be compensated by a change in the utility function. If π and $\tilde{\pi}$ are two beliefs with the same support, then in combination with u and $\tilde{u}$ as defined below, they represent the same preferences. For all $s \in S$ and $a \in A$,[12]

$$\tilde{u}(s, a) = \frac{\pi(s)}{\tilde{\pi}(s)}\, u(s, a).$$

Indeed, one can check that $E_{\pi,u}(f) = E_{\tilde{\pi},\tilde{u}}(f)$ for every act f. Abandoning the assumption of state-independent utilities, we thus lose the ability to infer subjective probabilities from preferences over acts. The only remnant of belief identification we can still hope to determine is the set of null states, and even that only works fully if the utility function is assumed to be non-constant in every state. Karni and Schmeidler (2016) showed how the uniqueness of the belief can be achieved for state-dependent utilities by enriching the framework. Let us stress that the representation in Savage's theorem is only unique (up to positive affine transformations of the utility function) assuming that utilities are state-independent. In fact, from any state-independent representation, one can construct state-dependent ones as above.

Let us see an example where assuming state-independent utilities leads to false conclusions. Suppose you participate in a medical study for a drug that reduces the intensity of all experiences to half of their previous level. The participants are randomly assigned to a treatment group (getting the drug) and a control group (getting a placebo) of equal size. You thus assign

[11] One way to defend being greedy despite all this goes as follows. We have already argued that probably something fishy is going on. Perhaps the entire experiment is a set-up by a decision theorist who wants to show that she can get people to lose faith in causality by giving them enough misleading evidence. Not so with you. All those people before you have probably been hired by the decision theorist and their choices and outcomes have been arranged a priori. So you will be better off being greedy following the usual causal reasoning.

[12] We use the convention that $\frac{0}{0} = 0$.

probability 50% to having received the drug. I know that you participated in some study but do not want to ask for specifics to respect your privacy. Instead, I try to figure out your subjective probability of having received a drug by asking you to rank two bets.

$$f = \begin{cases} \$0 & \text{if you have not received the drug} \\ \$30 & \text{if you have received the drug} \end{cases} \qquad g = \$10 \text{ in either case}$$

You are indifferent between both bets since the drug reduces the enjoyment you get from any amount of money by half. Not knowing the effect of this specific drug, I wrongly conclude from the indifference that you assign probability 33% to having received a drug. If the drug has no effect other can halving the intensity of your experiences, your preferences over bets will be perfectly consistent with Savage's axioms and thus admit a state-independent expected utility representation, but one that is based on the wrong belief and utility function.

The lesson to draw from this is that state-independent utilities may be ill-equipped for handling situations where the decision-maker's ability for experiences depends on the state. The situation of an observer who tries to infer a decision-maker's belief when the decision-maker's "utility generation" depends on the state "[...] is analogous to an astronomer who observes the stars, but does not know how their motion affects the operation of his own telescope" (Gilboa, 2009, p. 130).

5.4 Exercises

Exercise 5.1 (Independence of de Finetti's axioms). Show that all of F1–F5 are needed in Theorem 5.3. That is, show that for any of the axioms, there is a relation that satisfies these four axioms but violates the fifth.

Exercise 5.2 (Necessity of the Anscombe and Aumann axioms). Prove the "if part" of Theorem 5.9. That is, prove that if $\succsim$ is a relation on $\mathcal{L}(A)^S$ such that $\succsim$ is represented by $E_{\pi,u}$ for some belief $\pi \in \mathcal{L}(S)$ and linear utility function $u\colon \mathcal{L}(A) \to \mathbb{R}$, then $\succsim$ satisfies AA1–AA5.

Exercise 5.3 (Independence of the Anscombe and Aumann axioms). Show that the axioms in Anscombe and Aumann's characterization of subjective expected utility are logically independent. That is, for each of AA1–AA5, there is a relation on $\mathcal{L}(A)^S$ that violates this axiom but satisfies the remaining four.

Exercise 5.4 (Null states). Let $\succsim$ be a preference relation on A^S. That is, $\succsim$ satisfies S1. Show that the union of any two events that are null for $\succsim$ is also null for $\succsim$.

Exercise 5.5 (Non-trivial preferences over constant acts). Let $\succsim$ be a relation on A^S. Show that if $\succsim$ satisfies S1, S3, and S5, then there are $a, b \in A$ with $a \succ b$.

Exercise 5.6 (The Monty Hall problem). Suppose you participate in a game show and can choose between three doors. Behind one door is a car and behind the other two doors are goats. You have to pick one door. After that, the game show host (who knows where the car is) will open one of the remaining two doors revealing a goat. Then she gives you the opportunity to exchange your door for the other remaining one. Should you switch?

We make the following assumption.

(i) You want to win the car.

(ii) Prior to your first choice, you belief that each door is equally likely to contain the car.

(iii) If you choose the door with the car, the host is equally likely to open each of the two remaining doors.

Solve this problem by modeling it in Savage's framework.

Exercise 5.7 (Non-atomicity and convexity). Let π be a non-atomic (countably additive) probability measure on S. Show that π is convex.

Exercise 5.8 (Qualitative probabilities). Let $\succsim$ be a relation on A^S satisfying S1–S6. Let $\succsim^*$ be the corresponding qualitative probability. Without using Theorem 5.19, prove that $\succsim^*$ satisfies S6*.

6 Uncertainty Aversion

An experiment by Ellsberg (1961), known as the Ellsberg paradox, suggests that decision-makers prefer known risks to unknown risks. That is, uncertainty is less bad if the probabilities for events are known than if they are unknown.[13] This effect is called uncertainty aversion or ambiguity aversion. After discussing the Ellsberg paradox, we show how non-additive probabilities, so-called capacities, can be used to model uncertainty aversion. We follow Schmeidler (1989), who characterized uncertainty averse preferences in the model of Anscombe and Aumann as those of expected utility maximizers whose beliefs are non-additive probabilities. His result uses a weakening of the independence axiom of Anscombe and Aumann but retains the remaining axioms.

6.1 The Ellsberg Paradox

There are two urns, I and II, each containing 100 balls. Each ball is either red or black. Urn I contains 50 balls of each color, and there is no additional information about urn II. From each urn, one ball is chosen uniformly at random, which gives the following events.

IR The ball chosen from urn I is red.

IB The ball chosen from urn I is black.

IIR The ball chosen from urn II is red.

IIB The ball chosen from urn II is black.

For each of the four events, there is a bet that pays \$100 if the event occurs and \$0 otherwise. Ellsberg (1961) found that most decision-makers are indifferent between bets IR and IB and between IIR and IIB. In fact, many decision-makers are indifferent between all four bets. However, a non-negligible fraction prefers either of IR and IB to either of IIR and IIB.

One can translate this experiment into the framework of Anscombe and Aumann. The set of states consists of all possible distributions of balls in urn II. Since each distribution is determined by the number of red balls, we may let $S = \{0, 1, \ldots, 100\}$ be the set of states. The set of alternatives is $A = \{w, \ell\}$, where w means winning \$100 and ℓ means winning \$0. The probability of IR and IB is independent of the state. Hence, these two events correspond to the constant act, say f, that gives the uniform lottery $\frac{1}{2}\, w + \frac{1}{2}\, \ell$ over A for all states. The acts corresponding to IIR and IIB, denoted by g_{IIR} and g_{IIB}, are defined by

$$g_{\mathrm{IIR}}(s) = \frac{s}{100}\, w + \frac{100-s}{100}\, \ell \quad \text{and} \quad g_{\mathrm{IIB}}(s) = \frac{100-s}{100}\, w + \frac{s}{100}\, \ell$$

for all $s \in S$.

A preference relation $\succsim$ with $f \succ g_{\mathrm{IIR}}$ and $f \succ g_{\mathrm{IIB}}$ cannot be expressed by subjective expected utility maximization. One way to see this is by observing that these preferences

[13] The case of unknown probabilities is sometimes referred to as Knightian uncertainty.

violate the independence axiom AA2. Independence with $\alpha = \frac{1}{2}$ applied twice implies

$$f \succ \frac{1}{2} f + \frac{1}{2} g_{\mathrm{IIR}} \succ \frac{1}{2} g_{\mathrm{IIB}} + \frac{1}{2} g_{\mathrm{IIR}}.$$

But for all $s \in S$, we have

$$\frac{1}{2} g_{\mathrm{IIR}}(s) + \frac{1}{2} g_{\mathrm{IIB}}(s) = \frac{1}{2}\left(\frac{s}{100} w + \frac{100-s}{100} \ell + \frac{100-s}{100} w + \frac{s}{100} \ell\right) = \frac{1}{2} w + \frac{1}{2} \ell = f(s),$$

so that $\frac{1}{2} g_{\mathrm{IIR}} + \frac{1}{2} g_{\mathrm{IIB}} = f$. These two conclusions are contradictory. Another way so see it is by assuming that $\succsim$ is represented by a belief π and a utility function u with $u(w) > u(\ell)$. If the expected number of red balls under π is at least 50, then $g_{\mathrm{IIR}} \succsim f$. Similarly, if the expected number of black balls is at least 50, then $g_{\mathrm{IIB}} \succsim f$. So either $g_{\mathrm{IIR}} \succsim f$ or $g_{\mathrm{IIB}} \succsim f$.

To accommodate preferences as above, Schmeidler (1989) considers the framework of Anscombe and Aumann with non-additive probability measures, called capacities, instead of additive probabilities.

6.2 Capacities and Choquet Integration

Let S be a set. Recall that a probability measure on S is a non-negative (countably-additive) set function π on S with $\pi(S) = 1$ (which implies that $\pi(\emptyset) = 0$). Choquet capacities generalize the notion of probability by weakening additivity to monotonicity.

Definition 6.1 (Capacities)**.** Let $\nu\colon 2^S \to \mathbb{R}_+$ be a set function. Then, ν is a capacity (or non-additive probability) on S if

(i) $\nu(\emptyset) = 0$ and $\nu(S) = 1$, and

(ii) for all $E, F \subset S$ with $E \subset F$, $\nu(E) \leq \nu(F)$.

We denote by $\mathcal{C}(S)$ the set of capacities on S. One can define a notion of integration with respect to a capacity, that is, define the expected value of a real-valued function with respect to a capacity. For a function $f\colon S \to \mathbb{R}$, we write $\{f \geq x\} = \{s \in S\colon f(s) \geq x\}$ for short.

Definition 6.2 (Choquet integration)**.** Let ν be a capacity on S and $f\colon S \to \mathbb{R}$. Then, the Choquet integral of f with respect to ν is

$$\oint f d\nu = \int_{-\infty}^{0} (\nu(\{f \geq x\}) - 1)\, dx + \int_{0}^{\infty} \nu(\{f \geq x\}) dx,$$

where the integral on the right-hand side is the familiar Riemann integral, provided it is defined.

Observe that the first summand on the right-hand side vanishes if $f \geq 0$. In fact, if f is bounded below, say, $f \geq -c$ for $c \in \mathbb{R}_+$, then, exchanging variables with $y = x + c$ for the third equality, we get

$$\begin{aligned}
\oint f d\nu &= \int_{-c}^{0} (\nu(\{f \geq x\}) - 1)\, dx + \int_{0}^{\infty} \nu(\{f \geq x\}) dx \\
&= -c + \int_{-c}^{0} \nu(\{f + c \geq x + c\}) dx + \int_{0}^{\infty} \nu(\{f + c \geq x + c\}) dx \\
&= -c + \int_{0}^{\infty} \nu(\{f + c \geq y\}) dy \\
&= -c + \oint (f + c) d\nu.
\end{aligned}$$

If $f\colon S \to \mathbb{R}_+$ takes finitely many different values, say, $x_1 < \cdots < x_n$, then, with $x_0 = 0$,

$$\oint f d\nu = \sum_{k=1}^{n} (x_k - x_{k-1}) \nu(\{f \geq x_k\}).$$

Unlike the Riemann or Lebesgue integral, the Choquet integral is not linear. That is, for two functions $f, g\colon S \to \mathbb{R}$, it may be that

$$\oint f d\nu + \oint g d\nu \neq \oint (f + g) d\nu.$$

Example 6.3 (Non-additivity of the Choquet integral)**.** Let $S = \{0, 1\}$, $\nu(\{0\}) = \nu(\{1\}) = \frac{1}{3}$, $f(0) = g(1) = 1$, and $f(1) = g(0) = 0$. Then,

$$\begin{aligned}
&\oint f d\nu = \int_{0}^{1} \nu(\{0\}) dx = \frac{1}{3} = \int_{0}^{1} \nu(\{1\}) = \oint g d\nu, \text{ and} \\
&\oint (f + g) d\nu = \oint 1 d\nu = \int_{0}^{1} \nu(\{0, 1\}) dx = 1.
\end{aligned}$$

The Choquet integral does however satisfy three properties that are implied by linearity. To state those, we say that f and g are comonotone if $(f(s) - f(s'))(g(s) - g(s')) \geq 0$ for all $s, s' \in S$. That is, f and g increase and decrease together.

For all $f, g\colon S \to \mathbb{R}$ with $f \leq g$, $\oint f d\nu \leq \oint g d\nu$ (monotonicity)

For all $f\colon S \to \mathbb{R}$ and $c \in \mathbb{R}_{\geq 0}$, $\oint c f d\nu = c \oint f d\nu$ (positive homogeneity)

For all comonotone $f, g\colon S \to \mathbb{R}$, $\oint f d\nu + \oint g d\nu = \oint (f + g) d\nu$ (comonotone additivity)

6.3 Choquet Expected Utility

Consider the model of Anscombe Aumann with a finite set of states S and a finite set of alternatives A. For a capacity $\nu \in \mathcal{C}(S)$ and a linear utility function $u\colon \mathcal{L}(A) \to \mathbb{R}$, the Choquet expected utility of an act $f \in \mathcal{L}(A)^S$ is

$$\mathrm{E}_{\nu,u}(f) = \oint (u \circ f) d\nu.$$

Definition 6.4 (Choquet expected utility representation)**.** A relation $\succsim$ on $\mathcal{L}(A)^S$ admits a Choquet expected utility representation if there is a capacity $\nu \in \mathcal{C}(S)$ and a linear function $u\colon \mathcal{L}(A) \to \mathbb{R}$ so that $\mathrm{E}_{\nu,u}$ represents $\succsim$.

We have seen in Section 6.1 that a non-negligible fraction of decision-makers violates the independence axiom of Anscombe Aumann (AA2). Moreover, we have observed in Example 6.3 that Choquet expected utility can accommodate some violations of independence. Schmeidler (1989) showed that weakening independence to apply only to comonotone acts but keeping all the other axioms of Anscombe and Aumann captures precisely Choquet expected utility preferences.

For a relation $\succsim$ on $\mathcal{L}(A)^S$ and two acts $f, g \in \mathcal{L}(A)^S$, we say that f and g are comonotonic if there are no $s, s' \in S$ with $f(s) \succ f(s')$ and $g(s') \succ g(s)$, where we make the usual identification of lotteries with constant acts.

Definition 6.5 (Comonotonic independence)**.** A relation $\succsim$ on $\mathcal{L}(A)^S$ satisfies comonotonic independence if for all pairwise comonotonic $f, g, h \in \mathcal{L}(A)^S$ and $\alpha \in (0,1)$,

$$f \succsim g \text{ if and only if } f\alpha h \succsim g\alpha h.$$

Theorem 6.6 (Choquet expected utility representation, Schmeidler, 1989)**.** *A relation $\succsim$ on $\mathcal{L}(A)^S$ satisfies AA1, AA3, AA4, AA5, and comonotonic independence if and only if it admits a Choquet expected utility representation with a non-constant utility function.*

The proof is based on three lemmas.

Lemma 6.7 (Reduction to de Finetti acts)**.** *Let $\succsim$ be a relation on $\mathcal{L}(A)^S$ that satisfies AA1, AA3, AA4, AA5, and comonotonic independence. Then, there is linear function $u\colon \mathcal{L}(A) \to [0,1]$ and a relation $\tilde{\succsim}$ on $[0,1]^S$ satisfying AA1, AA3, monotonicity, and comonotonic independence so that*

$$f \succsim g \quad \textit{if and only if} \quad u \circ f \mathrel{\tilde{\succsim}} u \circ g.$$

Proof. First, we construct u. Observe that any two constant acts are comonotonic. Hence, the restriction of $\succsim$ to constant acts satisfies the independence axiom (vNM2). Since $\succsim$ satisfies AA1 and AA3, its restriction constant acts also satisfies vNM1 and vNM3. It follows from Theorem 4.6 that there is a linear utility function $u\colon \mathcal{L}(A) \to \mathbb{R}$ so that for all $p, q \in \mathcal{L}(A)$, $p \succsim q$ if and only if $u(p) \geq u(q)$. Since A is finite, u is bounded, and since $\succsim$ is non-trivial by AA5 and satisfies monotonicity (AA4), u is non-constant. We may thus assume that $\min\{u(p) \colon p \in \mathcal{L}(A)\} = 0$ and $\max\{u(p) \colon p \in \mathcal{L}(A)\} = 1$.

Observe that for all $\tilde{f} \in [0,1]^S$, there is $f \in \mathcal{L}(A)^S$ so that $\tilde{f} = u \circ f$. Let $\tilde{\succsim}$ be the relation on $[0,1]^S$ defined by letting

$$\tilde{f} \mathrel{\tilde{\succsim}} \tilde{g} \quad \text{if and only if} \quad f \succsim g,$$

where $f, g \in \mathcal{L}(A)^S$ with $\tilde{f} = u \circ f$ and $\tilde{g} = u \circ g$. AA4 implies that if $u \circ f = u \circ f'$, then $f \sim f'$. Hence, $\tilde{\succsim}$ is well-defined.

The fact that $\succsim$ satisfies AA1 and AA3 implies that $\tilde{\succsim}$ satisfies AA1 and AA3. Moreover, the monotonicity axiom AA4 implies that $\tilde{\succsim}$ is monotonic. Lastly, we observe that $\tilde{\succsim}$ satisfies comonotonic independence. Suppose $\tilde{f}, \tilde{g}, \tilde{h}$ are pairwise comonotonic. Then, if $f, g, h \in \mathcal{L}(A)^S$

are so that $\tilde{f} = u \circ f$, $\tilde{g} = u \circ g$, and $\tilde{h} = u \circ h$, f, g, h are pairwise comonotonic by definition of u. Comonotonic independence of $\succsim$ implies that independence holds for f, g, h, and thus, by linearity of u, also for $\tilde{f}, \tilde{g}, \tilde{h}$. This concludes the proof. $\square$

Definition 6.8 (Comonotonic linearity)**.** A function $F\colon [0,1]^S \to \mathbb{R}$ is comonotonically linear if

for all comonotonic $f, g \in [0,1]^S$, $\alpha \in [0,1]$, $F(\alpha f + (1-\alpha)g) = \alpha F(f) + (1-\alpha)F(g)$, and

for all $f \in [0,1]^S$, $c > 0$, $F(cf) = cF(f)$.

Lemma 6.9 (Representation by a comonotonically linear function)**.** *Let $\succsim$ be a relation on $[0,1]^S$ that satisfies AA1, AA3, AA4, AA5, and comonotonic independence. Then, there is a comonotonically linear function $F\colon [0,1]^S \to \mathbb{R}$ that represents $\succsim$.*

Proof. We split up $[0,1]^S$ into sets of comonotonically independent acts. Let $S = \{s_1, \dots, s_n\}$ and denote by Σ_n the set of all permutations of $\{1, \dots, n\}$. For $\sigma \in \Sigma_n$, let $A_\sigma = \{f \in [0,1]^S \colon f(s_{\sigma(1)}) \geq f(s_{\sigma(2)}) \geq \dots \geq f(s_{\sigma(n)})\}$. Observe that any two acts in A_σ are conomonotnic. Hence, the restriction of $\succsim$ to A_σ, denoted by $\succsim_\sigma$, satisfies AA2. Thus, $\succsim_\sigma$ satisfies vNM2, and by AA1 and AA3, it satisfies vNM1 and vNM3. Moreover, A_σ is convex. Hence, it follows from Theorem 4.6 that there is a linear function $u_\sigma\colon A_\sigma \to \mathbb{R}$ representing $\succsim_\sigma$. As in the proof of Lemma 6.7, we may assume that $\min\{u_\sigma(f)\colon f \in A_\sigma\} = 0$ and $\max\{u_\sigma(f)\colon f \in A_\sigma\} = 1$. Monotonicity of $\succsim$ (AA4) thus gives that $u_\sigma(\mathbf{0}) = 0$ and $u_\sigma(\mathbf{1}) = 1$.

Now define $F\colon [0,1]^S \to \mathbb{R}$ by letting $F(f) = u_\sigma(f)$ for all $f \in A_\sigma$ and $\sigma \in \Sigma_n$. To prove that F is well-defined, we need to verify that $u_\sigma(f) = u_{\sigma'}(f)$ whenever $f \in A_\sigma \cap A_{\sigma'}$. Observe that both u_σ and $u_{\sigma'}$ represent $\succsim$ restricted to $A_\sigma \cap A_{\sigma'}$. Moreover, $\mathbf{0}, \mathbf{1} \in A_\sigma \cap A_{\sigma'}$ and $A_\sigma \cap A_{\sigma'}$ is convex. Hence, the uniqueness part of Theorem 4.6 implies that u_σ and $u_{\sigma'}$ agree on $A_\sigma \cap A_{\sigma'}$. The fact that F is comonotonically linear follows from the fact that each u_σ is linear. $\square$

Lemma 6.10 (Representation by Choquet integration)**.** *Let $F\colon [0,1]^S \to \mathbb{R}$ be comonotonically linear with $F(\mathbf{1}) = 1$. Then, there exists a capacity ν on S so that for all $f \in [0,1]^S$,*

$$F(f) = \oint f d\nu.$$

Proof. We start by defining ν. For every $E \subset S$, let $\nu(E) = F(\mathbf{1}_E)$, where $\mathbf{1}_E$ is the indicator function of E. We prove by induction on the number of different values that f takes that F equals Choquet integration with respect to ν.

To this end, let $f = \sum_{l \leq k} \alpha_l \mathbf{1}_{E_l}$, where $\alpha_1, \dots, \alpha_k \in [0,1]$ and $\{E_1, \dots, E_k\}$ is a partition of S. If $k = 1$, say, $f = \lambda \mathbf{1}_S$ with $\lambda \in [0,1]$, then

$$F(f) = F(\lambda \mathbf{1}_S) = \lambda F(\mathbf{1}_S) = \lambda \nu(S) = \lambda \oint \mathbf{1} d\nu = \oint \lambda \mathbf{1} d\nu = \oint f d\nu.$$

For the induction step, assume the statement holds for all functions that take at most $k-1$ different values. For $l = 1, \dots, k$, let $x_l = f(s_l)$, where $s_l \in E_l$ and assume without loss of generality that $x_1 < \dots < x_k$. Define $\underline{g}, \overline{g} \in [0,1]^S$ by letting $\underline{g}(s) = \overline{g}(s) = f(s)$ for $s \in S \setminus E_2$,

$\underline{g}(s) = x_1$ and $\overline{g}(s) = x_3$ for $s \in E_2$. Then, $\underline{g}$ and $\overline{g}$ both take values $x_1, x_3, \ldots, x_k$ and are comonotonic. Hence, for $\alpha \in [0,1]$ with $x_2 = \alpha x_1 + (1-\alpha)x_3$, the fact that F and Choquet integration are conomotonically linear and the induction hypothesis imply that

$$F(f) = F(\underline{g}\alpha\overline{g}) = \alpha F(\underline{g}) + (1-\alpha)F(\overline{g}) = \alpha \oint \underline{g} d\nu + (1-\alpha)\oint \overline{g} d\nu = \oint (\underline{g}\alpha\overline{g}) d\nu = \oint f d\nu.$$

□

Proof of Theorem 6.6. Theorem 6.6 follows directly from Lemma 6.7, Lemma 6.9, and Lemma 6.10. □

6.4 Exercises

Exercise 6.1 (Properties of Choquet integration). Prove that Choquet integration satisfies monotonicity, positive homogeneity, comonotone additivity.

Exercise 6.2 (Independence of the Choquet expected utility axioms). Show that all of AA1, AA3, AA4, AA5, and comonotonic independence are needed in Theorem 6.6. That is, show that for each of the axioms, there is a relation that violates this axiom but satisfies the remaining four.

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